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Tests of the Stochastic Volatility with Jumps Model Driven by Moment Swaps

Ako Doffou
Department of Finance, School of Business, Shantou University
Shantou, Guangdong, China
E-mail: adoff21@gmail.com
Tel: + 86-754-86502882; Fax: +86-754-86503442

Abstract

This paper tests the pricing accuracy and the hedging performance of the stochastic volatility with random jumps model in markets extended to contain swap contracts whose payoffs depend on the realized higher moments of the state variable. Using a two-step iterative approach, latent model variables are first filtered and then used to estimate the model parameters. The tests on European options and variance swaps written on the S&P 500 index show superior pricing accuracies in-sample and out-of-sample and jump risk is priced. Hedging strategies involving higher-order moment swaps perform better across all moneyness and maturity classes.

Keywords: Stochastic volatility; Random jumps; Variance swaps; Higher-order moment swaps; Self-financing portfolio.
JEL Classifications: C52, G12, G13,

1. Introduction

Moment swaps are derivatives whose payoff depends on the realized higher moments of the underlying state variable. This payoff depends on the powers of the daily log-returns and allows moment swaps to provide protection against various types of supply and demand shocks in capital markets. Variance swaps are created in the case of squared log-returns. Variance swaps are today liquidly traded, driven by different types of state variables and offer protection against the volatility regime fluctuations. In addition to the variance, skewness, kurtosis and higher-order moments play important roles in the distribution of asset prices. Higher-order moment derivatives can be useful to protect against inaccurately estimated higher moments such as skewness and kurtosis. Doffou (2019), Rompolis and Tzavalis (2017), and Schoutens (2005) show that the classical hedge of the variance swap in terms of a position in a log-contract and a dynamic trading strategy can be significantly enhanced by using third-order moment swaps.

Recent studies suggest that power-jump assets are the natural choice to complete the market. For instance, an incomplete Levy market where power-assets of any order can be traded will yield a complete market. Power assets and realized higher moments are highly linked and they are virtually the same in a discrete time framework (Corcuera *et al*. 2005).

There has been a proliferation of studies extending the Black-Scholes (1973) option pricing model. But only a few of these studies addressed hedging contingent claims under more general assumptions about the state variable stochastic process. Well known examples of such studies are the stochastic volatility (SV) model of Heston (1993), the SV and random jumps model (SVJ) of Bates (1996), and the SV, stochastic interest rates and random jumps model of Doffou and Hilliard (2001).

Other studies have focused on the pricing formula for moment swaps. For example, Zhu and Lian (2011) proposed a closed-form solution to the variance swap under the Heston model. Zheng and Kwok (2014) derived the moment-generating function (MGF) for the Heston model with simultaneous jumps in the asset price and variance processes. Pun *et* al. (2015) obtained the MGF for models with mean reversion in asset price, multi-factor stochastic volatility and simultaneous jumps in prices and volatility factors. Using delta hedging strategies in an incomplete market cannot lead to a perfect hedge against jump risk and volatility risk linked to a position in contingent claims.

Locally risk-minimizing delta hedging strategies which attempt to hedge the option contract using only the state variable and minimizing the variance of the cost process of a non-self-financed hedging position are not adequate. In the presence of jumps, these strategies perform very poorly like the classic delta hedging strategies (Tankov *et al*. 2007). Other hedging strategies use option contracts to reduce or eliminate volatility risk (Bakshi *et al*. 1997) or protect against jump risk (Coleman *et al*. 2006; Cheang *et al*. 2015). Cross hedging strategies or delta-vega hedging can totally remove an option contract exposure to volatility risk but not its exposure to jump risk. Jump risk can be hedged either by using a risk-minimization strategy (Coleman *et al*. 2006; Tankov *et al*. 2007) or by discretizing jump sizes to compute the hedge ratios of the other options (Utzet *et al*. 2002). But these two methodologies are limited. First, they do not characterize the maturity and moneyness of the option contracts to be used to efficiently and effectively set up the hedge. Further, these two approaches do not pick up the options' exposure to volatility risk. Finally, Empirical evidences show that these two methods do not outperform the delta hedging strategy (Cheang *et al*. 2015).

The model considered here assumes jumps to occur in the price process. However, the literature has already provided solution to correlated jumps between the asset return and its volatility. Such a well-received consideration is being tested in a separate paper. This paper ***tests*** empirically the stochastic volatility with jumps (SVJ) model of Bates (1996) within the framework of Rompolis and Tzavalis (2017) which offers a different approach to hedge derivatives under more general assumptions of the state variable process. Under this approach, perfect hedging strategies of contingent claims under stochastic volatility and random jumps can be achieved by extending the market to contain swap contracts whose payoffs depend on the realized higher moments of the state variable price process. Hence, volatility and jump risks are hedged simultaneously without the need to rely on the well-known risk minimization criterion. Most specifically, a derivative price exposure to stochastic volatility and random jumps can be effectively hedged by including in the self-financing portfolio variance swap contracts and higher-order moment swap contracts respectively. Jumps size is random and therefore enough higher-order moment swaps are needed to achieve a perfect hedge. In the limit, as the number of higher-order moment swaps increases to infinity, the value of the self-financing hedging portfolio converges to the value of the derivative, making the market quasi complete in the spirit of Bjork *et al*. (1997) and Jarrow and Madan (1999).

Constructing perfect hedging strategies under an incomplete market model has been addressed by a few studies using new assets. The market can be completed if power jump assets are added (Corcuera *et al.* 2005). But, the prices of power jump assets are not observable in the market and therefore cannot be traded (Olhede *et al*. 2010). Even though the prices of higher-order moment swaps are not directly available and observable in the market, higher-order moment swaps are directly affected by the state variable price changes and so <u>can be observed and traded</u> in capital markets. Consequently, it makes sense to evaluate hedging strategies driven by higher-order moment swaps. The model ***to be tested*** in this paper extends the above listed studies in many ways. *First*, it accounts for stochastic volatility and random jumps. Hence, both volatility risk and jump risk are hedged. To hedge both risks requires the use of a new swap contract called the bipower variation swap which helps distinguish a variance swap exposure to jump risk and volatility risk. *Second*, hedging is extended to option pricing using the Black-Scholes methodology. The implementation of these hedging strategies in a market enlarged with higher-order moment swaps leads to specific option prices. *Finally*, the performance of the proposed hedging strategies is assessed using real data.

In practice, higher-order moment swaps are not always available and illiquidity and trading costs are important considerations. Counterparty risk must also be factored in when more financial derivatives are purchased for hedging. This paper provides an evidence for the importance of considering stochastic volatility, random jumps, and higher-order moment swaps in the pricing and hedging model. The empirical results obtained here show that adding jump components to a stochastic volatility model in a market enlarged with higher-order moment swaps leads to a more realistic modeling of conditional higher moments as well as the moneyness and maturity effects, an improvement of the modeling of the term structure of the conditional variance, and a superior model pricing performance. A perfect hedge of derivative securities can be achieved when the state variable follows the stochastic volatility with random jumps model in a market enlarged with higher-order moment swaps. The key contribution of this paper is that hedging strategies, driven by the stochastic volatility with random jumps model in a market enlarged with higher-order moment swaps, perform better across all moneyness and maturity classes.

This paper is organized as follows. Section 2 defines moment swaps. Section 3 introduces the pricing and hedging model. The model parameters are estimated in Section 4. Section 5 tests the in-sample and the out-of-sample pricing accuracy as well as the hedging performance of the model. Finally, Section 6 concludes the paper.

2. Moment Swaps

Suppose a liquidly traded asset (stock or stock index) with a continuous dividend yield $y \geq 0$ has a price process modelled by an Ito semi-martingale $P = \{P_t,\ t \geq 0\}$ such that $P > 0$ and $P_- > 0$. In addition to this asset, a bond or money market account with a constant compound interest rate α is available with a price process $\Theta = \{\Theta_t = e^{\alpha t}, t \geq 0\}$. Consider n equally spaced time intervals of length Δt such that $t_j = j\Delta t$, with $j = 0, 1, 2, \ldots., n$ and $\Psi = n\Delta t$ is the expiration date of the derivative contract written on the state variable price P_t. The price of the state variable at each interval j is denoted P_j for simplicity. In practice, the t_j are the daily closing times and P_j is the closing price at day j. It follows that the daily log-returns are given by $\ln(P_j) - \ln(P_{j-1})$, $j = 1, 2, \ldots., n$.

Assume futures contracts written on the state variable price P exist with expiration date Ψ. By risk-neutral valuation, the futures contract price process follows $F_t = P_t exp(\alpha - y)(\Psi - t)$. For simplification, the futures price at the discrete time t_j is denoted F_j. The mth-moment swap on the stock is a contract in which the two counterparties agree to exchange at maturity a nominal amount multiplied by the difference between a fixed level contract price and the realized level of the mth-order non-central sample moment of the log-return over the life of the contract. The payoff function is defined by

$$MS_P^m = NA \sum_{j=1}^{n} \ln\left(\frac{P_j}{P_{j-1}}\right)^m , \tag{1}$$

where NA is the nominal or notional amount and n is the number of segments of length Δt within the time interval $[0, \Psi]$, $t \in [0. \Psi]$.

For $= 2$, equation 1 gives the expression of the $2nd$-moment swap or variance swap. Variance swaps are basically forward contracts in which the counterparties agree to exchange a notional amount multiplied by the difference between a fixed variance and the realized variance. The fixed variance is the variance swap rate or the variance forward price. Variance swaps offer protection against volatility shocks. The $3rd$- moment swap is linked to the realized skewness and offers protection against changes in the symmetry of the underlying distribution. Changes in the tail behavior of the underlying distribution created by the occurrences of unexpected large jumps are shielded by the $4th$-moment swap related to the realized kurtosis. If the futures price is the state variable driving the moment derivatives, then the payoff function of the mth-moment swap on the futures is

$$MS_F^m = NA\sum_{j=1}^n \ln\left(\frac{F_j}{F_{j-1}}\right)^m \quad (2)$$

Using the link between the stock and futures prices, $F_j = P_j exp(\alpha - y)(\Psi - t_j)$, and setting the notional amount NA to one, the relationship between the futures and the stock moment swaps is derived as follows

$$MS_F^m = \sum_{j=1}^n \left[-(\alpha - y)\Delta t + \ln\left(\frac{P_j}{P_{j-1}}\right)\right]^m = \sum_{j=1}^n \sum_{h=0}^m \binom{m}{h}(-(\alpha - y)\Delta t)^h \ln\left(\frac{P_j}{P_{j-1}}\right)^{m-h} =$$
$$\sum_{h=0}^{m-2}\binom{m}{h}(-(\alpha - y)\Delta t)^h MS_P^{(m-h)} - (\alpha - y)\Psi(-(\alpha - y)\Delta t)^{m-1} + m(-(\alpha - y)\Delta t)^{m-1}\Omega_P$$

where $\Omega_P = \sum_{j=1}^n \ln\left(\frac{P_j}{P_{j-1}}\right) = \ln\left(\frac{P_\Psi}{P_0}\right) = \ln(P_\Psi) - \ln(P_0)$.

The term Ω_P is the *log-contract* on the stock and plays a critical role in the hedging of moment swaps. If Δt to the higher order powers is very small and therefore negligible, the above listed expression can be approximated to

$$MS_F^m \approx MS_P^m - m(\alpha - y)\Delta t MS_P^{(m-1)} \quad (3)$$

Let $(\mathcal{H}, \mathcal{L}, \mathcal{P})$ be a filtered probability space where $\mathcal{L} = \{\mathcal{L}_t, t \geq 0\}$ is the filtration which satisfies the usual conditions and $\mathcal{P}$ the physical probability measure. In this economy, the dynamics of the log-price process $\chi = \ln P$ satisfies Assumption 1 of Ait-Sahalia and Jacod (2009). Hence, as the time interval gets closer to zero, the terminal values of higher-order moment swaps in discrete time converge to their continuous time values. This convergence justifies an analysis in continuous time of these terminal values defined in discrete time.

The quadratic variation of the log-price process χ in the time interval $[0, \Psi]$ is defined as $\langle\chi, \chi\rangle_{0,\Psi} \in \mathcal{N}^*(\mathcal{P})$ where $\mathcal{N}^*(\mathcal{P})$ is the $\mathcal{N}^*$ norm on the physical probability measure $\mathcal{P}$. The power variation of χ at the mth order in the interval $[0, \Psi]$ is defined by $\sum_{0<t\leq\Psi}(\Delta\chi_t)^m \in \mathcal{N}^*(\mathcal{P})$, for $m \geq 2$. This establishes the existence of continuous time swap contract rates and their related expected returns. In the absence of arbitrage, there is a risk-neutral probability measure (equivalent martingale measure) $\mathfrak{P}$, continuous with respect to the physical probability measure $\mathcal{P}$, under which $\langle\chi, \chi\rangle_{0,\Psi}$ and $\sum_{0<t\leq\Psi}(\Delta\chi_t)^m$ can be priced, that is $\langle\chi, \chi\rangle_{0,\Psi} \in \mathcal{N}^*(\mathfrak{P})$ and $\sum_{0<t\leq\Psi}(\Delta\chi_t)^m \in \mathcal{N}^*(\mathfrak{P})$ for $m \geq 2$.

In continuous time, the variance swap payoff given by equation (1) when $m = 2$ converges in probability to the annualized quadratic variation of the log-price process which is $\frac{1}{\Psi}\langle\chi, \chi\rangle_{0,\Psi}$ (Protter, 1990). Therefore, under the equivalent martingale measure $\mathfrak{P}$, the variance swap rate in continuous time, MSR_t^2, is the risk-neutral expected value of $\frac{1}{\Psi}\langle\chi, \chi\rangle_{0,\Psi}$ given by

$$MSR_t^2 = E_t^{\mathfrak{P}}\left(\frac{1}{\Psi}\langle\chi, \chi\rangle_{0,\Psi}\right) \quad (4)$$

The payoff, in continuous time when n grows to infinity, of the mth-order moment swap given in equation (1) for $m \geq 3$ converges in probability to $\frac{1}{\Psi}\sum_{0<t\leq\Psi}(\Delta\chi_t)^m$. Consequently, under the equivalent martingale measure $\mathfrak{P}$, the continuous time mth-order moment swap rate at time t, MSR_t^m, is given by

$$MSR_t^m = E_t^{\mathfrak{P}}\left(\frac{1}{\Psi}\sum_{0<t\leq\Psi}(\Delta\chi_t)^m\right) \quad (5)$$

The variation of the discontinuous part of the log-price process χ which is $\frac{1}{\Psi}\sum_{0<t\leq\Psi}(\Delta\chi_t)^2$ cannot be captured in the higher-order moment swap defined in equation (5). The annualized quadratic variation of the log-price process $\frac{1}{\Psi}\langle\chi, \chi\rangle_{0,\Psi}$ which contains the variation of this discontinuous part can be in fact decomposed into a continuous part and a discontinuous part as follows

$$\frac{1}{\Psi}\langle\chi,\chi\rangle_{0,\Psi} = \frac{1}{\Psi}\langle\chi,\chi\rangle_{0,\Psi}^{c} + \frac{1}{\Psi}\sum_{0<t\leq\Psi}(\Delta\chi_t)^2, \tag{6}$$

with $\langle\chi,\chi\rangle^c$ being the continuous part of $\langle\chi,\chi\rangle$.

The decomposition given in equation (6) above has been proved in Barndorff-Nielsen and Shephard (2004), Ait-Sahalia and Jacod (2009), and Rompolis and Tzavalis (2017). Given the above decomposition, a new swap contract called *bipower variation swap* was introduced in Rompolis and Tzavalis (2017) to price the discontinuous part of $\langle\chi,\chi\rangle$. In discrete time, the terminal value at time Ψ of the bipower variation swap takes the following expression assuming a notional amount of one:

$$MS_{P,\Psi}^2 = \left(\sum_{j=1}^{n}\ln\left(\frac{P_j}{P_{j-1}}\right)^2 - \frac{\pi}{2}\left(\frac{n}{n-1}\right)\sum_{j=1}^{n-1}\left|\ln\frac{P_j}{P_{j-1}}\right|\left|\ln\frac{P_{j+1}}{P_j}\right|\right). \tag{7}$$

The volatility risk premium and the jump risk premium which both affect the variance swap rate can now be analyzed separately using the bipower variation swap.
The bipower variation swap rate at time t in continuous time is given by

$$MSR_{t,bip}^2 = E_t^{\mathfrak{P}}\left(\frac{1}{\Psi}\sum_{0<t\leq\Psi}(\Delta\chi_t)^2\right), \tag{8}$$

$$\text{with } MSR_t^2 - MSR_{t,bip}^2 = E_t^{\mathfrak{P}}\left(\frac{1}{\Psi}\langle\chi,\chi\rangle_{0,\Psi}^{c}\right). \tag{9}$$

Based on equation (9), the terminal value of a portfolio composed of a long position in a variance swap contract and a short position in a bipower variation swap contract is a function of only the continuous component of the quadratic variation of χ . Hence, the market price of volatility risk can be assessed given the value of this portfolio.

3. Pricing and Hedging Model Driven by Higher-Order Moment Swaps

The higher-order moment swap contracts introduced in the previous paragraph can be used to price and hedge derivative securities which include European options, barrier options, volatility swaps, and volatility swap options. The state variable which here is the stock price is assumed to follow the stochastic volatility with jumps (SVJ) model of Bates (1996). Jumps do occur due to supply and demand shocks in the stock market. The occurrence of jumps causes the distribution of the spot price to be more skewed and kurtotic than the lognormal but does not affect the risk-neutralized expectation. The model can accommodate small and large jumps as well as the frequent and infrequent arrival of information in the stock market. The stochastic evolution of the instantaneous conditional volatility, a randomization known to induce excess kurtosis, directly affects contingent claims pricing biases. The proposed Bates model is part of a family of models with independent fat-tailed shocks to the stock price. Accordingly, the dynamics of the stock price P_t and that of its variance $\mho_t$ follow the stochastic processes

$$\frac{dP_t}{P_t} = (\vartheta_t^P - \omega\bar{\vartheta})dt + \sqrt{\mho_t}\mathrm{d}Z_{1,t} + k_t\mathrm{d}q_t \tag{10}$$

$$d\mho_t = \beta(\delta - \mho_t)dt + \phi\sqrt{\mho_t}dZ_{2,t}, \tag{11}$$

where $\vartheta_t^P = \alpha - y + \lambda P\mho_t + \omega(\bar{\vartheta} - \bar{\vartheta}^{\mathfrak{P}})$; λP is the market price of risk; k_t is the random percentage jump conditional upon a Poisson-distributed event occurring, where $1 + k_t$ is log-normally distributed: $\ln(1 + k_t) \sim N(\ln(1 + \bar{k}) - 0.5\varepsilon^2, \varepsilon^2) = N(\vartheta k, \phi_k^2)$, with $\mathrm{E}(k_t) = \bar{k}$; ω is the frequency of Poisson events; ε is the jump dispersion parameter; q_t is a Poisson counter with intensity ω : $\mathrm{Prob}(dq_t = 1) = \omega dt$ and $\mathrm{Prob}(dq_t = 0) = 1 - \omega dt$; $\bar{\vartheta} = \exp\left(\vartheta k + \frac{1}{2}\phi_k^2\right) - 1$ and $\bar{\vartheta}^{\mathfrak{P}}$ is the risk-adjusted mean value of ϑ. The stock price process is like the geometric Brownian motion process most of the time, but on average ω times per period the price jumps discretely by a random percentage. Jump random variables are uncorrelated, i.e., $(dq, k) = 0$, $Cov(dZ_1, dq) = Cov(dZ_1, k) = 0$.

Because the increments to a standard Brownian motion dZ_1 and dZ_2 are assumed to be correlated with correlation coefficient ξ, i.e., $Cov(dZ_1, dZ_2) = \xi$, there is a third Brownian motion process Z_3 independent of Z_1 such that $dZ_{2,t} = \xi dZ_{1,t} + \sqrt{1-\xi^2} dZ_{3,t}$. The stochastic discount factor process $\mathcal{D}$ for the proposed model can be expressed as follows

$$\frac{d\mathcal{D}_t}{\mathcal{D}_t} = -(\alpha + \omega \bar{\vartheta}_{\mathcal{D}})dt - \lambda P \sqrt{\mho_t} dZ_{1,t} - \frac{\lambda \mho \sqrt{\mho_t}}{\phi \sqrt{1-\xi^2}} dZ_{3,t} + k_{\mathcal{D},t} dq_t, \tag{12}$$

where $\lambda\mho$ picks up the market price of volatility risk, $\bar{\vartheta}_{\mathcal{D}}$ is the mean jump size given by $\bar{\vartheta}_{\mathcal{D}} = exp\left(\vartheta k_{\mathcal{D}} + \frac{1}{2}\phi_{k_{\mathcal{D}}}^2\right) - 1$ and $\ln\left(1 + k_{\mathcal{D},t}\right) \sim N\left(\vartheta k_{\mathcal{D}}, \phi_{k_{\mathcal{D}}}^2\right)$. The mean jump size $\bar{\vartheta}_{\mathcal{D}}$ is constrained to zero in the spirit of Pan (2002) and Broadie *et al.* (2007) which leads to a constant value for the jump frequency parameter ω under the risk-neutral probability measure $\mathfrak{P}$. The closed-form solutions of the rates of these swaps and the expected value changes of positions in them at time zero are provided in Rompolis and Tzavalis (2017).

In the absence of arbitrage, the variance swap rate at time $t \in [0, \Psi]$ under the stochastic volatility and jump model is expressed as follows

$$MSR_t^2 = \frac{1}{\Psi}\left(\int_0^t \mho_v dv + \int_0^t \tilde{k}_v^2 \, dq_v + A(t, \mho_t) + B(t)\right), \tag{13}$$

where $A(t, \mho_t) = \eta\left(\zeta_t \mho_t + (1 - \zeta_t)\delta^{\mathfrak{P}}\right)$; $\zeta_t = \left(1 - e^{-\eta\beta^{\mathfrak{P}}}\right) / \left(\eta\beta^{\mathfrak{P}}\right)$; $\delta^{\mathfrak{P}}$ and $\beta^{\mathfrak{P}}$ are respectively the risk-neutralized values of δ and β ; $\eta = \Psi - t$; and $B(t) = \omega\eta\left(\vartheta_2^{\mathfrak{P}}\right)$, with $\vartheta_2^{\mathfrak{P}}$ being the non-central second-order moment of $\tilde{k}$ under the risk-neutral probability measure $\mathfrak{P}$. The expected change in value of a long position in the variance swap at time zero under the physical probability measure $\mathcal{P}$ is given by $E_t^{\mathcal{P}}\left[d\mho_t^{MSR^2}\right]$, with $\mho_t^{MSR^2} = e^{-\alpha\eta}(MSR_t^2 - MSR_0^2)$ and further expressed as follows

$$\vartheta_t^{MSR^2} dt \equiv E_t^{\mathcal{P}}\left[d\mho_t^{MSR^2}\right] = \left(\alpha \mho_t^{MSR^2} + \frac{\partial \mho^{MSR^2}}{\partial \ln \mho} \tilde{\lambda}\mho + \frac{\omega e^{-\alpha\eta}}{\Psi}\left(\vartheta_2 - \vartheta_2^{\mathfrak{P}}\right)\right) dt, \tag{14}$$

where $\tilde{\lambda}\mho = \phi\xi\lambda P + \lambda\mho$, ϑ_2 is the second-order non-central moment of $\tilde{k}$ under the physical probability measure $\mathcal{P}$, and $\vartheta_2^{\mathfrak{P}}$ the risk-neutral measure of ϑ_2.

Equation (14) clearly shows that the expected change in value of a long position in a variance swap at time zero depends on a jump-component risk premium, the price of volatility risk $\lambda\mho$, and the market price of risk λP. The jump components are jumps related to the underlying stock price P. The expected excess value change $\vartheta_t^{MSR^2} - \alpha\mho_t^{MSR^2}$ is negative because $\vartheta_2 - \vartheta_2^{\mathfrak{P}}$ is negative and Carr and Wu (2010) showed that $\tilde{\lambda}\mho$ is negative. The negative sign of $\vartheta_2 - \vartheta_2^{\mathfrak{P}}$ is due to $\vartheta_k^{\mathfrak{P}} < \vartheta_k < 0$ and $\phi_k^2 < \left(\phi_k^{\mathfrak{P}}\right)^2$, with $\vartheta_k^{\mathfrak{P}}$ and $\phi_k^{\mathfrak{P}}$ being respectively the risk-neutral measures of ϑ_k and ϕ_k (Broadie *et al.* 2007). This is consistent with the fact that the variance swap pays when the stock price P suddenly decreases or increases because of jumps that occur due to supply and demand shocks in the stock markets. To protect against these unexpected supply and demand fluctuations, risk averse investors are willing to pay a premium to take long positions in the variance swap.

Similarly, in the absence of arbitrage and under the stochastic volatility model, the bipower variation swap rate (when $m = 2$) and the higher-order moment swap rate (when $m \geq 3$) are given by

$$MSR_t^m = \frac{1}{\Psi}\left(\int_0^t \tilde{k}_v^m \, dq_v + \omega\eta\vartheta_m^{\mathfrak{P}}\right), \tag{15}$$

while the expected change in value at time t of a long position in these contracts at time zero is

$$\vartheta_t^{MSR^m} dt \equiv E_t^{\mathcal{P}}\left[d\mho_t^{MSR^m}\right] = \left(\alpha \mho_t^{MSR^m} + \frac{\omega e^{-\alpha\eta}}{\Psi}\left(\vartheta_m - \vartheta_m^{\mathfrak{P}}\right)\right) dt, \tag{16}$$

where ϑ_m is the mth-order non-central moment of $\tilde{k}$ under the physical probability measure $\mathcal{P}$ and $\vartheta_m^{\mathfrak{P}}$ is the risk-neutralized value of ϑ_m.

Equation (16) indicates that the expected excess value change of a long position in a bipower variation swap, given by $\vartheta_t^{MSR^2} - \alpha \mho_t^{MSR^2}$, is negative because $\vartheta_2 - \vartheta_2^{\mathfrak{P}}$ is negative. In addition to being negative, the expected value change is smaller in magnitude than that of the variance swap because the bipower variation swap pays only when the stock price exhibits some jumps during the time segment $[0, \Psi]$ while the variance swap also picks up any increase in the spot variance $\mho$. Further, equation (16) shows that the expected excess value change of a long position in higher-order moment swaps, $\vartheta_t^{MSR^m} - \alpha \mho_t^{MSR^m}$ with $m \geq 3$, has the sign of the difference $\vartheta_m - \vartheta_m^{\mathfrak{P}}$. Given the normality assumption of the log-jump size $\tilde{k}$, ϑ_m and $\vartheta_m^{\mathfrak{P}}$ exist for all values of m. In general, for higher-order moment swaps, the expected excess value change is strictly negative for even values of m and strictly positive for odd values of m, that is

$$\vartheta_t^{MSR^m} - \alpha \mho_t^{MSR^m} < 0, \text{ if } m \text{ is even}; \vartheta_t^{MSR^m} - \alpha \mho_t^{MSR^m} > 0, \text{ if } m \text{ is odd}. \tag{17}$$

At time t, the derivative security $\mathfrak{S}$ to be priced and hedged under the stochastic volatility and jump model has a price $\mathfrak{S}_t$ which depends on time t, the underlying stock price P_t and the spot variance $\mho_t$, that is $\mathfrak{S}_t = \mathfrak{S}(t, P_t, \mho_t)$ with $0 \leq t \leq \Psi$. The function $\mathfrak{S}(.)$ is assumed to be continuous with partial derivatives of any order. The self-financing portfolio which replicates the price of the derivative security $\mathfrak{S}_t$ is assumed to be composed of the underlying stock, the money market account, the variance swaps, the bipower variation swaps, and the higher-order moment swaps. As a result, the implied vector of hedge ratios at time t is characterized by the quantity of the state variable P, the position in the bond or money market account Θ, the number of long positions in variance, bipower variation and higher-order moment (up to order m) swaps. Hence, the implied vector of hedge ratio at time t, given the various positions taken at time zero, is expressed by $\mathrm{H}_t = \left(\mathrm{H}_t^P, \mathrm{H}_t^\Theta, \mathrm{H}_t^{MSR^2}, \mathrm{H}_t^{MSR_t^2}, \dots\dots\dots, \mathrm{H}_t^{MSR_t^m}\right)'$. To be able to replicate the contingent claim $\mathfrak{S}$, it is necessary to have an adequate finite but sufficiently large number M of higher-order moment swaps such that for all $t \in [0, \Psi]$

$$\mathfrak{S}_t = \mathrm{H}_t^P P_t + \mathrm{H}_t^\Theta \Theta_t + \mathrm{H}_t^{MSR^2} \mho_t^{MSR^2} + \lim_{M\to\infty} \sum_{m=2}^{M} \mathrm{H}_t^{MSR_t^m} \mho_t^{MSR_t^m}, \tag{18}$$

with the hedge ratios given by $H_t^P = \frac{\partial \mathfrak{S}}{\partial P}$,

$$\mathrm{H}_t^\Theta = \Theta_t^{-1}\left(\mathfrak{S}_t - \mathrm{H}_t^P P_t - \mathrm{H}_t^{MSR^2} \mho_t^{MSR^2} - \lim_{M\to\infty} \sum_{m=2}^{M} \mathrm{H}_t^{MSR_t^m} \mho_t^{MSR_t^m}\right),$$

$$\mathrm{H}_t^{MSR^2} = e^{\alpha\eta} \frac{\partial \mathfrak{S}}{\partial MSR^2},$$

$$\mathrm{H}_t^{MSR_t^2} = e^{\alpha\eta}\left(\frac{\Psi}{2}\left(\frac{\partial^2 \mathfrak{S}}{\partial \ln P^2} - \frac{\partial \mathfrak{S}}{\partial \ln P}\right) - \frac{\partial \mathfrak{S}}{\partial MSR^2}\right) \text{ and}$$

$$\mathrm{H}_t^{MSR_t^m} = e^{\alpha\eta}\left(\frac{\Psi}{m!}\right)\left(\frac{\partial^m \mathfrak{S}}{\partial \ln P^m} - \frac{\partial \mathfrak{S}}{\partial \ln P}\right), \text{ for } m \geq 3.$$

The expected return at time t of the derivative security $\mathfrak{S}$ is given in Rompolis and Tzavalis (2017) as

$$E_t^{\mathcal{P}}\left[\frac{d\mathfrak{S}_t}{\mathfrak{S}_t}\right] = \alpha dt + \frac{\partial \ln \mathfrak{S}}{\partial \ln P}(\vartheta_t^P + y - \alpha)dt + \frac{\partial \ln \mathfrak{S}}{\partial \mho^{MSR^2}}\left(\vartheta_t^{MSR^2} - \alpha \mho_t^{MSR^2}\right)dt$$
$$+ e^{\alpha\eta}\left(\frac{\Psi}{2}\left(\frac{\partial^2 \mathfrak{S}/\mathfrak{G}}{\partial \ln P^2} - \frac{\partial \ln \mathfrak{S}}{\partial \ln P}\right) - \frac{\partial \ln \mathfrak{S}}{\partial MSR^2}\right)\left(\vartheta_t^{MSR_t^2} - \alpha \mho_t^{MSR_t^2}\right)dt$$
$$+ \lim_{M\to\infty} \sum_{m=3}^{M} e^{\alpha\eta}\left(\frac{\Psi}{m!}\right)\left(\frac{\partial^m \mathfrak{S}/\mathfrak{G}}{\partial \ln P^m} - \frac{\partial \ln \mathfrak{S}}{\partial \ln P}\right)\left(\vartheta_t^{MSR_t^m} - \alpha \mho_t^{MSR_t^m}\right)dt. \tag{19}$$

Equation (19) shows that the derivative security $\mathfrak{S}$ can be replicated in a market extended to contain variance swaps, bipower variation swaps and a number M of higher-order moment swaps. The delta hedged gains of the self-financing portfolio will converge to zero if M is large enough to make the market quasi complete, leading to the existence of a unique risk-neutral measure $\mathfrak{P}$ under which the derivative $\mathfrak{S}_t$ can be priced (Jarrow and Madan 1999; Bjork *et al*. 1997). The price of the contingent claim $\mathfrak{S}_t$ can be derived by either adopting the Bates (1996) equilibrium approach or by following the Rompolis and Tzavalis (2017) methodology in eliminating all the stochastic terms in $\mathfrak{S}_t$. In the absence of jumps ($\omega = 0$), equation (19) shows that taking a position in the underlying stock, the money market account and the variance swap contract, the derivative $\mathfrak{S}$ can be perfectly replicated.

The contingent claim exposure to volatility risk is hedged by the positions $\mathrm{H}_t^{MSR^2}$ held in variance swaps. But variance swaps are also exposed to jump risk and the position $\mathrm{H}_t^{MSR_t^2}$ held in the bipower variation swap adjusts the change in value of the derivative $\mathfrak{S}_t$ due to changes in the stock price as a result of the exposure to the same jump risk. The exposure of the contingent claim $\mathfrak{S}_t$ to jump risk is hedged by higher-order moment swaps. For all values of m, the hedge ratios $\mathrm{H}_t^{MSR_t^m}$ are the same for call and put options of the same exercise price and maturity date for the put-call parity technical arbitrage condition to hold.

4. Data

Spot data on the S&P 500 index as well as real data on variance swap rates and European call and put option prices written of the S&P 500 index are used in this study. The option data set is from the OptionMetrics Ivy data base and spans from January 2003 to December 2017 for a total coverage of fifteen years. This data set is used to test the pricing accuracy of the model and to evaluate the relative performance of various hedging strategies.

Daily closing option quotes from the Chicago Board Options Exchange are used each Wednesday and mi-quotes are computed as averages of the bid and ask quotes. The underlying index level is adjusted for dividends and then matched with each option quote. The adjustment for dividends is carried out by subtracting from the index level the present value of the future realized stream of dividends between the quote date and the maturity date of the option. For a given option maturity, the risk-free rate is computed via interpolation of available T-bill rates. To effectively carry out the empirical tests, only at-the-money (ATM) and out-of-the money (OTM) calls and puts are used because they are more actively traded than in-the-money (ITM) options. Options with less than one week to maturity are excluded from the sample. Option prices less than 3/8 are too close to tick size to reflect true option values and are therefore deleted from the sample. Option contracts with zero open interest, with extreme moneyness and those that violate various boundary and no-arbitrage conditions are discarded. The series of dividend yield y is derived from the sample and estimated using the put-call parity technical arbitrage condition applied to the at-the-money (ATM) European options. The data for the variance swap rates written on the S&P 500 index provided by a major broker-dealer are daily closing quotes of variance swap rates traded at the over-the-counter market with maturity intervals η of 1, 2, 3, 6, 9, 12, and 24 months from January 1, 2003 to December 31, 2017. These variance rate data are sampled weekly on every Wednesday to avoid the impact of weekday patterns on the estimation of the parameters. The quotes from the previous business day is used if a given Wednesday is a holiday. This classification generates 784 weekly observations for each series. Prices that reflect the illiquidity of the variance swaps market are deleted from the sample. The resulting descriptive statistics appear in Table 2.

The filtered option data set is summarized in Table 1 with a total of 36,164 call option contracts and 35,988 put option contracts. Panels A - C in Table 1 are arranged over six moneyness (P/E)

Table 1: S&P 500 Index Call and Put Option Data 1/2003–12/2017

	$TTM < 30$		$30 < TTM < 90$		$90 < TTM < 180$		$TTM > 180$		All	
	Panel A: Number of Call and Put Option Contracts									
	Call	**Put**	**Call**	**Put**	**Call**	**Put**	**Call**	**Put**	**Call**	**Put**
$P/E < 0.975$	539	-	3,771	-	3,098	-	3,682	-	11,090	-
$0.975 < P/E < 1$	1,159		3,103		1,086		901		6,249	
$1 < P/E < 1.025$	1,231		2,595		937		723		5,486	
$1.025 < P/E < 1.05$	941		1,851		735		474		4,001	
$1.05 < P/E < 1.075$	692		1,389		602		414		3,097	
$P/E > 1.075$	1,250		2,236		1,539		1,216		6,241	
All	**5,812**		**14,945**		**7,997**		**7,410**		**36,164**	**35,988**
	Panel B: Average Call and Put Prices									
$P/E < 0.975$	3.57	-	10.39	-	17.12	-	31.10	-	18.65	-
$0.975 < P/E < 1$	9.75		22.14		34.11		56.74		27.22	
$1 < P/E < 1.025$	20.13		32.39		41.09		56.83		34.46	
$1.025 < P/E < 1.05$	32.26		41.86		48.13		62.55		43.58	
$1.05 < P/E < 1.075$	43.11		52.68		55.20		67.26		53.14	
$P/E > 1.075$	56.18		62.47		64.12		72.59		63.57	
All	**28.86**		**32.39**		**37.35**		**48.09**		**36.18**	
	Panel C: Average Implied Volatility from Call and Put Options									
$P/E < 0.975$	16.26	-	16.10	-	16.31	-	16.95	-	16.45	-
$0.975 < P/E < 1$	15.99		17.12		17.36		18.20		17.12	
$1 < P/E < 1.025$	17.52		17.97		17.83		17.99		17.86	
$1.025 < P/E < 1.05$	19.48		18.87		18.51		18.15		18.87	
$1.05 < P/E < 1.075$	22.86		19.99		19.04		18.61		20.26	
$P/E > 1.075$	32.47		22.54		19.81		18.93		23.16	
All	**21.23**		**18.32**		**17.74**	.	**17.73**		**18.56**	

Notes: The sample is composed of European call and put options on the S&P 500 Index. Closing quotes each Wednesday in the whole period running from January 2003 to December 2017 are used. Moneyness and maturity filters applied here include among others Bakshi et al. (1997). The implied volatilities are derived using Black-Scholes (1973). Only the call option data and prices are reported here. The put prices can simply be obtained using the put-call parity technical arbitrage condition.

Table2: Variance Swap Rates Descriptive Statistics

Maturity (months)	Mean	Standard Deviation	Skewness	Excess Kurtosis	Weekly Autocorrelation
1	20.457	6.527	0.724	0.733	0.948
2	20.544	6.288	0.749	0.756	0.969
3	20.641	6.024	0.683	0.639	0.977
6	21.237	5.844	0.711	0.821	0.981
9	21.615	5.729	0.642	0.478	0.983
12	21.991	5.621	0.579	0.147	0.986
24	22.592	5.480	0.530	-0.195	0.989

Notes: Table 2 shows the mean, standard deviation, skewness, excess kurtosis, and weekly autocorrelation of the variance swap rate quotes in volatility percentage points on the S&P 500 Index at different maturities in months. Weekly data taken every Wednesday from 1/1/2003 to 12/31/2017 generated 784 observations for each series.

categories and four categories in time to maturity (TTM) in days. The number of contracts in each category is reported in Panel A, the average call price in each category is reported in Panel B, and the average Black-Scholes implied volatility in each category is reported in Panel C. The average implied volatility computed from the data set in Panel C, with maturity intervals of 1, 2, 3, 6, 9, 12, and 24 months, is needed to estimate the parameters of the stochastic volatility with jumps model and to calculate the third-order risk neutral moment $\vartheta_3^{\mathfrak{P}}$. Once the call price is known, the put price can be obtained using the put-call parity relation. Hence, only the call option data and prices are reported in Table 1 to make the table more readable. Each column of Panel C shows the evidence of the volatility smirk across moneyness.

5. Empirical Investigations

Two different investigations are conducted. The first assesses the pricing accuracy of the stochastic volatility with random jumps model (SVJ) in a market enlarged with higher-order moment swaps. The second tests the relative performance of various hedging strategies using real data on variance swap rates and European call and put prices.

5.1. Testing the Stochastic Volatility with Random Jumps Model

5.1.1. Model Parameters Estimation

A two-step iterative approach is used to estimate the model parameters. The intuition driving this approach is to first filter latent model variables and then use these variables to estimate the model parameters. These two steps are repeated until there is no further improvement in the aggregate objective function. In a way, this approach is a modification of the implicit parameter estimation put forward in Bates (2000). This is the first time the two-step iterative approach has been improved to simultaneously estimate the model's structural parameters not related to jumps, the model's structural parameters related to jumps and the spot volatility.

The implementation of the SVJ model poses the challenge of jointly estimating the model's structural parameters not related to jumps, $\Gamma = \{\beta, \delta, \phi, \xi\}$, the model structural parameters related to jumps or vector of unknown jump parameters, $\Upsilon = [\sigma_P, \bar{k}^{\mathfrak{P}}, \omega^{\mathfrak{P}}, \varepsilon]$, and the spot volatilities $\{\mho_t\}$. The unknown jump parameter σ_P is the instantaneous variance of the stock price conditional on no jumps. The model structural parameters (both jump related and non-jump related) and the spot volatilities are estimated using option and return data in a two-step iterative approach, an improvement of the approach adopted in Christoffersen *et al*. (2009) to account for jumps.

Consider a sample of option data covering $\mathcal{T}$ Wednesdays. In the implementation of the model, a full calendar year of option data are used and so $\mathcal{T} = 52$. Given a set of initial values for Γ, Υ, and $\{\mho_t\}$, a two-step iterative procedure is then initiated. In the *first step*, solve $\mathcal{T}$ sum of squared pricing error optimization problems, given a set of model structural parameters Γ and Υ, as follows.

Table 3: Parameter Estimates and Option Fit to the Stochastic Volatility with Jumps Model

	Parameter Estimates								IVRMSELF		
Year	β	δ	ϕ	ξ	σ_P	$\bar{\vartheta}^{\mathfrak{P}}$	$\omega^{\mathfrak{P}}$	ε	**In-sample**	**Out-of-sample**	**Number of Obs**
2003	1.2133	0.0367	0.6473	-0.5047	0.2198	0.0714	0.8429	0.1001	1.7119	N/A	5,389
2004	1.5036	0.0274	0.3688	-0.6056	0.1548	0.0571	0.6743	0.0802	0.8177	0.9424	5,608
2005	1.5802	0.0233	0.4997	-0.5935	0.1281	0.0005	0.5057	0.0605	0.5385	1.1256	6,309
2006	1.6652	0.0158	0.5746	-0.6129	0.1281	0.0786	0.9271	0.1103	1.9832	2.0036	6,748
2007	2.7494	0.0143	0.7694	-0.6990	0.1817	0.0007	0.5058	0.0604	0.6122	0.7875	8,103
2008	1.5550	0.0118	0.8351	-0.7781	0.3331	-0.0801	0.6745	0.0408	1.1651	1.5645	8,865
2009	1.9722	0.0184	0.6132	-0.5907	0.3148	0.0643	0.7586	0.0906	0.5886	0.7873	2,938
2010	1.3446	0.0328	0.4709	-0.5373	0.2255	0.0572	0.6746	0.0801	1.0003	1.0502	4,176
2011	1.1361	0.0637	0.5655	-0.5398	0.2420	-0.0701	0.5902	0.0507	0.6729	0.7802	3,888
2012	1.3218	0.0675	0.5832	-0.6711	0.1780	0.0644	0.7588	0.0905	0.8245	1.0047	3,551
2013	1.5973	0.0422	0.6821	-0.6606	0.1423	0.0645	0.7584	0.0907	0.5227	1.0661	3,427
2014	2.3688	0.0350	0.5993	-0.6135	0.1418	0.0572	0.6748	0.0804	0.7001	0.7339	3,069
2015	2.0940	0.0336	0.5264	-0.6618	0.1667	0.0006	0.5055	0.0606	0.5788	0.6336	3,035
2016	1.0669	0.0429	0.4865	-0.5173	0.1583	0.0573	0.6747	0.0803	0.5863	0.6165	3,479
2017	0.9954	0.0288	0.4736	-0.5840	0.1109	0.0787	0.9273	0.1104	0.8174	0.9465	3,567
Total									**1.2476**	**1.3745**	**72,152**

Notes: The parameters of the stochastic volatility with random jumps model are estimated year by year based on joint options and returns data. Wednesday closing option quotes from Table 1 are used in the computations of the speed of mean reversion β, the long-term mean value δ of the variance of the price ($\mho_t$), the volatility of the variance (volatility of volatility) ϕ, and the correlation ξ between returns and return variance. Return data on the S&P 500 index are used in the computations of the jump-diffusion parameters which include the spot volatility σ_P, the mean jump size $\bar{\vartheta}^{\mathfrak{P}}$, the jump frequency parameter $\omega^{\mathfrak{P}}$, and the jump dispersion parameter ε. All the structural parameters reported above are estimated using the iterative two-step approach described in section 5. The in-sample root mean squared errors are calculated using the Black-Scholes Vega approximation to IVRMSELF.

$$\{\hat{\mho}_t\} = \arg\min \sum_{k=1}^{N_t} \left(\mathfrak{S}_{k,t} - \mathfrak{S}_k(\Gamma, \Upsilon, \mho_t)\right)^2 / \nu_{k,t}^2, t = 1,2, \ldots, \mathcal{T}, \tag{20}$$

where the market-observed price of contract k on day t is $\mathfrak{S}_{k,t}$ and the associated model price is $\mathfrak{S}_k(\Gamma, \Upsilon, \mho_t)$. The number of contracts available on day t is N_t while $\nu_{k,t}$ is the Vega of contract k at time t. The Vega $\nu_{k,t}$ is the sensitivity of the option calculated using the implied volatility from the market price $\mathfrak{S}_{k,t}$ of the option. In the *second step*, solve one aggregate sum of squared pricing error optimization problem, given a set of spot volatilities $\{\mho_t\}$ obtained in step 1, as follows:

$$\{\hat{\Gamma}, \hat{\Upsilon}\} = \arg\min \sum_{k,t}^{N} \left(\mathfrak{S}_{k,t} - \mathfrak{S}_k(\Gamma, \Upsilon, \mho_t)\right)^2 / \nu_{k,t}^2 , \tag{21}$$

where $N = \sum_{t=1}^{\mathcal{T}} N_t$.

The above two steps are repeated until there is no further improvement in the aggregate objective function in step 2. The scaling factor $1/\nu_{k,t}$ is the key difference between this procedure and the Bates (2000) method of implicit parameters estimation. Because of this scaling factor, the objective function is seen as an approximation to implied volatility errors. The model price of the option can then be taken as a first-order approximation of the market price of the option around the implied Black-Scholes volatility, that is:

Table 4: Option Fit to the Christoffersen *et al.* (2009) Two-Factor Stochastic Volatility Model

Year	**IVRMSELF**		**Number of Observations**
	In-Sample	**Out-Sample**	
2003	1.8361	N/A	5,389
2004	1.4693	1.9859	5,608
2005	1.3527	1.7064	6,309
2006	2.2145	2.2713	6,748
2007	2.1364	2.1694	8,103
2008	1.4227	1.7136	8,865
2009	0.6317	0.8144	2,938
2010	1.0439	1.0891	4,176
2011	1.1752	1.6245	3,888
2012	0.9791	1.0418	3,551
2013	0.5536	1.1347	3,427
2014	0.7011	0.8029	3,069
2015	0.7143	0.7358	3,035
2016	0.6018	0.6425	3,479
2017	0.8472	0.9633	3,567
Total	1.4368	1.5692	72152

Notes: Table 4 shows the computed implied volatility root mean squared error loss function (IVRMSELF), both in-sample and out-of-sample, for the two-factor stochastic volatility model of Christoffersen et al. (2009) fitted to the same S&P 500 Index call and put data from 01/2003 to 12/2017. The values of the IVRMSELF computed for the stochastic volatility with random jumps model in Table 3 are much lower than those computed for the two-factor stochastic volatility model of Christoffersen et al. (2009). This is the hard evidence that the stochastic volatility with random jumps model fits better the option data than the two-factor stochastic volatility model in a market enlarged with variance swaps.

Table 5: Regression Analysis for Pricing Errors for European S&P500 Call and Put Options

Regression Parameters	Call Options	Put Options
Intercept	**0.1284** (0.0114)	**0.1308** (0.0132)
P/E	**- 0.1027** (0.0113)	**- 0.1041** (0.0133)
TM	**0.0426** (0.0011)	**0.0475** (0.0014)
VOL	**0.0211** (0.0113)	**0.0212** (0.0132)
$Adj.R^2$	**0.0493**	**0.0512**
Number of Observations	**36,164**	**35,988.**

Notes: Pricing errors are regressed over moneyness (P/E), time to maturity (TM), and the previous day's annualized standard deviation (VOL) of the S&P 500 index returns. The regression is run separately for call and put options using the equation below:

$PPE_k(t) = b_0 + b_1\frac{P(t)}{E_k} + b_2TM_k + b_3VOL(t-1) + \mathcal{O}_k(t),\ \ k = 1, \dots, N.$

Standard errors are in parentheses and are computed using the White (1980) heteroskedasticity consistent estimator. The previous day's annualized standard deviations of the S&P 500 index returns are computed from 5-minute intraday returns.

$$\mathfrak{S}_k(\Gamma, \Upsilon, \mho_t) \approx \mathfrak{S}_{k,t} + \nu_{k,t}\left(\phi_{k,t} - \phi_{k,t}(\Gamma, \Upsilon, \mho_t)\right), \quad (22)$$

where the implied Black-Scholes volatilities from the observed market price and from the model price are respectively $\phi_{k,t}$ and $\phi_{k,t}(\Gamma, \Upsilon, \mho_t)$, and $\nu_{k,t}$ is the Vega of the option which measures the Black-Scholes sensitivity of the option price to changes in the volatility $\phi_{k,t}$. This approximation will be used in the next paragraph to assess the model fit.

5.1.2. Model Pricing Performance Assessment and Results

Using the approximation put forward in equation (22), the model fit can be assessed by the value of the *implied volatility root mean squared error loss function* (IVRMSELF) given by:

$$IVRMSELF \equiv \left(\frac{1}{N}\sum_{k,t}^{N}\left(\phi_{k,t} - \phi_{k,t}(\Gamma, \Upsilon, \mho_t)\right)^2\right)^{1/2} \approx \left(\frac{1}{N}\sum_{k,t}^{N}\left(\mathfrak{S}_{k,t} - \mathfrak{S}_k(\Gamma, \Upsilon, \mho_t)\right)^2/\nu_{k,t}^2\right)^{1/2}.$$

This approximation to the implied volatility errors also used in Christoffersen *et al*. (2009), Schwartz and Trolle (2009), Carr and Wu (2007), is very useful and less costly numerically in a large scale empirical estimation undertaken here.

Moneyness, maturity and volatility effects on pricing bias can be further examined using a regression analysis. The dependent variable is the percentage pricing error of a given option in the sample at a given date. The independent variables are the moneyness, the time to maturity and the volatility of the S&P 500 index return. The regression equation is given by

$$PPE_k(t) = b_0 + b_1\frac{P(t)}{E_k} + b_2TM_k + b_3VOL(t-1) + \mathcal{O}_k(t), k = 1, \dots\dots. N, \quad (23)$$

where $PPE_k(t)$ is the percentage pricing error of option k on date t; P/E_k and TM_k represent respectively the moneyness and time to maturity of the option contract; $VOL(t-1)$ stands for the previous day's annualized standard deviation of the S&P 500 index return; and $\mathcal{O}_k(t)$ is the error term. Because this is a cross-sectional regression, the standard errors (in parentheses in Table 5) are computed using the White (1980) heteroskedasticity consistent estimator. The regression is run separately for calls and puts and the results are summarized in Table 5. Each independent variable has statistically significant explanatory power of the remaining pricing errors for both call and put options. Consequently, the pricing errors for each option category have some maturity, intra-daily volatility and moneyness related biases with different magnitudes. The pricing errors have the same sign and therefore biased in the same direction. The pricing errors relative to the S&P 500 index's volatility on the previous day are negligible and practically stationary, confirming the importance of modeling stochastic volatility. The pricing errors for both call and put options reported confirm that modeling both stochastic volatility and jumps is important. The adjusted R^2 of 4.93% for call options and 5.12% for put options show that the collective explanatory power of these independent variables is quite low.

Table 3 shows the results of the parameter estimates and the stochastic volatility with random jumps model fit to the option data. Table 4 shows the Christoffersen *et al.* (2009) two-factor stochastic volatility model fit to the same data set. The tests of the stochastic volatility

Table 6: In-Sample IVRMSELF by Moneyness and Maturity, 1/2003–12/2017

	$TTM < 30$	$30 < TTM < 90$	$90 < TTM < 180$	$TTM > 180$	**All**
$P/E < 0.975$	1.2691	0.7753	0.6372	0.7538	0.7689
$0.975 < P/E < 1$	1.2487	0.6835	0.5926	0.6704	0.7931
$1 < P/E < 1.025$	1.1847	0.6639	0.6582	0.6947	0.8523
$1.025 < P/E < 1.05$	1.7645	0.7828	0.7963	0.7658	1.0948
$1.05 < P/E < 1.075$	2.1837	1.1069	0.8895	0.8394	1.2253
$P/E > 1.075$	2.9573	1.8684	1.3029	1.4257	2.1061
All	2.1541	0.9886	0.8419	0.8951	**1.2476**

Notes: The parameter estimates reported in Table 3 are used to compute the IVRMSELF for different degrees of moneyness and maturity. The values of the implied volatility root mean squared error loss function (IVRMSELF) computed are all lower than those reported in Christoffersen et al. (2009) for all degrees of moneyness and maturity.

Table 7: Results of the European Options Hedging Strategy $HS_{(P,MSR^2,MSR_t^{12})}$

		Mean Hedging Error $\overline{\mathcal{HE}}$			**Mean Absolute Error MAE**			**Root Mean Squared Error RMSE**		
Moneyness	**Hedging Strategy**	**30**	**90**	**180**	**30**	**90**	**180**	**30**	**90**	**180**
Panel A: European Call Options Hedging Errors										
$P/E = 1.00$	$HS_{(P,MSR^2,MSR_t^{12})}$	0.053	-0.013	-0.011	0.326	0.237	0.184	0.519	0.306	0.272
$P/E = 0.975$	$HS_{(P,MSR^2,MSR_t^{12})}$	0.037	-0.009	-0.008	0.314	0.213	0.175	0.468	0.317	0.245
$P/E = 0.95$	$HS_{(P,MSR^2,MSR_t^{12})}$	0.054	-0.029	-0.021	0.436	0.387	0.369	0.725	0.541	0.481
Panel B: European Put Options Hedging Errors										
$P/E = 1.10$	$HS_{(P,MSR^2,MSR_t^{12})}$	0.019	-0.026	-0.004	0.355	0.438	0.527	0.610	0.729	0.781
$P/E = 1.08$	$HS_{(P,MSR^2,MSR_t^{12})}$	0.031	-0.033	-0.003	0.469	0.527	0.578	0.642	0.751	0.796
$P/E = 105$	$HS_{(P,MSR^2,MSR_t^{12})}$	0.049	-0.012	-0.001	0.528	0.559	0.602	0.793	0.808	0.811
$P/E = 1.025$	$HS_{(P,MSR^2,MSR_t^{12})}$	0.036	-0.005	-0.003	0.648	0.619	0.534	0.997	0.742	0.729

Notes: The values of the mean hedging error $\overline{\mathcal{HE}}$, the mean absolute error MAE and the root mean squared error RMSE are displayed in this Table for the hedging strategy $\mathrm{HS}_{(\mathrm{P},\mathrm{MSR}^2\mathrm{MSR}_t^{12})}$ executed on European options written on the S&P 500 index. This strategy is driven by multiple instruments which include the bipower variation swap and mth-order moment swaps ($3 \leq m \leq 12$) in the replicating portfolio. The mth-order non-central risk-neutral moment at each point in time is used to approximate the mth-order moment swap rate. The deltas of the hedging instruments are drawn from the estimates of the model in Table 3. Maturities considered are 30, 90 and 180 days. Hedging errors are computed each day with daily rebalancing from 1/2003 to 12/2017.

with random jumps model are implemented using 2003-2017 data on European call and put options on the S&P 500 index. The data set is disaggregated into fifteen separate samples containing each one full year of option data from 2003 to 2017. Next, fifteen in-sample exercises are undertaken and then the first fourteen groups of parameter estimates are evaluated one-year out of sample. To get the out-of-sample results, the out-of-sample spot volatilities for the following year are calculated using the in-sample structural parameters $\{\Gamma, \Upsilon\}$ in Table 3 and the *first step* of § 5.1.1 for each except the final year of the sample. The overall sum of squared pricing errors is then computed as the aggregate of the 52 sums of squares from the first step. This out-of-sample methodology is consistent with Huang and Wu (2004). The speed of adjustment or speed of mean reversion of volatility parameter β estimate most often lies between 1.17 and 2.40, indicating that the half-life of variance shocks is within five to ten months. The long-term mean value of variance or unconditional variance δ is between 0.01 and 0.03. The volatility of variance or volatility of volatility ϕ is between 0.37 and 0.83 while the correlation **ξ** between the index returns and return variance is within – 0.78 and – 0.50. The jump diffusion parameter estimates in Table 3 include the instantaneous variance conditional on no jumps σ_P, the risk-neutralized mean jump size $\bar{\vartheta}^{\mathfrak{P}}$, the jump frequency parameter $\omega^{\mathfrak{P}}$, and the jump dispersion parameter **ε**. Estimates of σ_P range within 11.09% and 33.31% and those of $\bar{\vartheta}^{\mathfrak{P}}$ between 0.00 (virtually positive and negative jumps cancel out in 2005, 2007 and 2015) and – 0.08 (negative average largest jump in 2008). Estimates obtained for $\omega^{\mathfrak{P}}$ are between 0.50 (lower number of jumps in 2005, 2007 and 2015) and 0.93 (greater number of jumps in 2006 and 2017) while those obtained for ε range within 0.04 (year 2008) and 0.11 (in 2006 and 2017). The overall *in-sample* implied volatility root mean squared error loss function (IVRMSELF) numerical value computed is 1.2476% versus 1.4368% obtained using the same data set for the Christoffersen *et al*. (2009) two-factor stochastic volatility model. This represents an improved pricing error accuracy of 13.17%. The overall *out-of-sample* IVRMSELF computed is 1.3745% compared to 1.5692% computed using the same data set for the Christoffersen *et al.* (2009) two-factor stochastic volatility model. This represents an improved pricing accuracy of 12.41%. The out-of-sample comparison between the stochastic volatility with random jumps model tested here and the two-factor stochastic volatility model of Christoffersen *et al*. (2009) leads to the conclusion that the model with random jump processes in this paper most effectively contributes in modeling option data in a market enlarged with higher-order moment swaps. The results in Tables 3 and 6 show that the stochastic volatility with random jumps model captures more of the variability in the option data, allows for richer modeling of maturity and moneyness, and offers very rich patterns in the term structure of the conditional variance. These results are confirmed by an excellent empirical fit to the time series as well as to the cross-sectional dimension using a tractable and parsimonious stochastic volatility with random jumps model.

5.2. Testing the Relative Performance of Various Hedging Strategies

This section tests the relative performance of alternative hedging strategies using real data on variance swap rates and European call and put option prices. *First*, the well-known delta hedging strategy driven by one instrument, the underlying asset, is considered and called HS_P. At time t, a long position is taken on contingent claim $\mathfrak{S}$ with exercise price E and maturity η. To hedge this exposure, a short position in $\text{H}_t^P = \partial\mathfrak{S}/\partial P$ shares of the stock is taken and the residuals $R_t = \mathfrak{S}_t - \text{H}_t^P P_t$ is invested in the money market account. At the subsequent time $t + \Delta t$ where $\Delta t = 1/360$ for one-day rebalancing of the portfolio or $\Delta t = 1/(360 \times 288)$ for five-minute rebalancing, the hedging error $(\mathcal{HE})$ of this strategy is computed as:

$$\text{HS}_P(t + \Delta t) = (\mathfrak{S}_{t+\Delta t} - \mathfrak{S}_t) - (\text{H}_t^P(P_{t+\Delta t} - P_t) + \alpha R_t \Delta t + y\text{H}_t^P P_t \Delta t). \quad (24)$$

Second, a hedging strategy driven by two instruments, a variance swap and its underlying state variable which here is the stock price, is considered and referred to as $\text{HS}_{(P,MSR^2)}$.The variance swap is issued at time t and has the same maturity date η as the option contract, that is $\eta = \Psi$. A long position in the contingent claim $\mathfrak{S}$ at time t is delta-hedged by taking a short position in $\text{H}_t^P = \frac{\partial\mathfrak{S}}{\partial P}$ shares of the stock, a short position in $\text{H}_t^{MSR^2} = e^{\alpha\eta}\frac{\partial\mathfrak{S}}{\partial MSR^2}$ variance swaps and investing the residuals $R_t = \mathfrak{S}_t - \text{H}_t^P P_t - \text{H}_t^{MSR^2}\mho_t^{MSR^2}$ in the money market account. Because the swap is issued at time t, $\mho_t^{MSR^2} = 0$. The hedging error $(\mathcal{HE})$ of this strategy at time $t + \Delta t$ is computed as follows:

$$\text{HS}_{(P,MSR^2)}(t + \Delta t) = (\mathfrak{S}_{t+\Delta t} - \mathfrak{S}_t)$$
$$-\left(\text{H}_t^P(P_{t+\Delta t} - P_t) + \text{H}_t^{MSR^2}\left(\mho_{t+\Delta t}^{MSR^2} - \mho_t^{MSR^2}\right) + \alpha R_t \Delta t + y\text{H}_t^P P_t \Delta t\right). \quad (25)$$

The strategy $\mathrm{HS}_{(P,MSR^2)}$ is compared to a cross-hedge strategy to evaluate its relative performance in hedging volatility risk in the presence of jump risk. The underlying stock and another option contract named $\emptyset$ are used in this two-instrument hedging strategy. The vega neutrality of the portfolio is assured first using option $\emptyset$. Next, the portfolio is made delta-neutral by buying or selling the underlying stock (Bakshi *et* al. 1997). This cross-hedge strategy or delta-vega hedging is referred to as $\mathrm{HS}_{(P,\emptyset)}$.

Third, a hedging strategy driven by multiple instruments which include the bipower variation swap and mth-order moment swaps $(3 \leq m \leq 12)$ in the replicating portfolio is considered. This strategy is referred to as $\mathrm{HS}_{(P,MSR^2,MSR_t^{12})}$. The hedging error $(\mathcal{HE})$ of this strategy is given by:

$$\mathrm{HS}_{(P,MSR^2,MSR_t^{12})}(t+\Delta t) = (\mathfrak{S}_{t+\Delta t} - \mathfrak{S}_t)$$
$$-\left[\mathrm{H}_t^P(P_{t+\Delta t} - P_t) + \mathrm{H}_t^{MSR^2}\left(\mho_{t+\Delta t}^{MSR^2} - \mho_t^{MSR^2}\right) + \sum_{k=2}^{12} \mathrm{H}_t^{MSR_t^k}\left(\mho_{t+\Delta t}^{MSR_t^k} - \mho_t^{MSR_t^k}\right)\right]$$
$$-(\alpha R_t \Delta t + y\mathrm{H}_t^P P_t \Delta t), \tag{26}$$

where $R_t = \mathfrak{S}_t - \mathrm{H}_t^P P_t - \mathrm{H}_t^{MSR^2}\mho_t^{MSR^2} - \sum_{k=2}^{12} \mathrm{H}_t^{MSR_t^k}\mho_t^{MSR_t^k} = \mathfrak{S}_t - \mathrm{H}_t^P P_t$ because for all values of k, $\mho_t^{MSR^2} = \mho_t^{MSR_t^k} = 0$ as the swap inception date is time t.

Finally, to hedge against random jumps, a special case of the multi-instrument strategy is considered with $m = 3$, which contains only the third-order moment swap and is referred to as $\mathrm{HS}_{(P,MSR^2,MSR_t^3)}$. The hedging error $(\mathcal{HE})$ of this special strategy is computed as follows:

$$\mathrm{HS}_{(P,MSR^2,MSR_t^3)}(t+\Delta t) = (\mathfrak{S}_{t+\Delta t} - \mathfrak{S}_t) - \left[\mathrm{H}_t^P(P_{t+\Delta t} - P_t) + \mathrm{H}_t^{MSR^2}\left(\mho_{t+\Delta t}^{MSR^2} - \mho_t^{MSR^2}\right)\right]$$
$$-[\mathrm{H}_t^{MSR_t^3}\left(\mho_{t+\Delta t}^{MSR_t^3} - \mho_t^{MSR_t^3}\right) + \alpha R_t \Delta t + y\mathrm{H}_t^P P_t \Delta t] \tag{27}$$

The results of these various hedging strategies are summarized in Tables 7-8. The key contribution of this paper is empirical and is illustrated in Tables 7-8 which clearly shows that

Table 8: Results of Alternative European Options Hedging Strategies

		Mean Hedging Error $\overline{\mathcal{HE}}$			**Mean Absolute Error MAE**			**Root Mean Squared Error RMSE**		
Moneyness	**Hedging Strategy**	**30**	**90**	**180**	**30**	**90**	**180**	**30**	**90**	**180**
Panel A: European Call Options Hedging Errors										
$P/E = 1.00$	HS_P	-0.191	-0.094	-0.038	2.015	2.013	2.017	3.025	3.002	3.016
	$HS_{(P,MSR^2)}$	0.062	-0.017	-0.016	0.406	0.271	0.211	0.607	0.374	0.318
	$HS_{(P,MSR^2,MSR^3}$	0.059	-0.016	-0.015	0.398	0.264	0.202	0.581	0.367	0.305
	$HS_{(P,\emptyset)}$	-0.068	-0.019	-0.018	0.447	0.285	0.213	0.629	0.426	0.339
$P/E = 0.975$	HS_P	-0.124	-0.067	-0.027	2.161	2.191	2.231	3.046	3.131	3.211
	$HS_{(P,MSR^2)}$	0.051	-0.015	-0.013	0.428	0.256	0.208	0.582	0.417	0.298
	$HS_{(P,MSR^2,MSR^3}$	0.049	-0.012	-0.011	0.396	0.241	0.194	0.537	0.385	0.279
	$HS_{(P,\emptyset)}$	0.058	0.019	0.019	0.458	0.269	0.214	0.635	0.439	0.328
$P/E = 0.95$	HS_P	-0.125	-0.048	0.002	2.109	2.327	2.436	3.007	3.358	3.463
	$HS_{(P,MSR^2)}$	0.074	-0.041	-0.032	0.685	0.463	0.428	0.936	0.739	0.642
	$HS_{(P,MSR^2,MSR^3}$	0.067	-0.035	-0.027	0.648	0.421	0.395	0.902	0.713	0.604
	$HS_{(P,\emptyset)}$	0.095	0.048	0.039	0.762	0.578	0.469	1.054	0.847	0.709
Panel B: European Put Options Hedging Errors										
$P/E = 1.10$	HS_P	-0.155	-0.108	-0.068	0.839	1.017	1.217	1.463	1.614	1.873
	$HS_{(P,MSR^2)}$	-0.038	-0.039	-0.008	0.547	0.653	0.748	0.838	0.977	0.984
	$HS_{(P,MSR^2,MSR^3}$	0.027	-0.032	-0.007	0.538	0.624	0.719	0.822	0.965	0.978
	$HS_{(P,\emptyset)}$	-0.049	-0.048	-0.009	0.652	0.886	0.973	0.948	1.142	1.285

		Mean Hedging Error $\overline{\mathcal{HE}}$			Mean Absolute Error MAE			Root Mean Squared Error RMSE		
Moneyness	**Hedging Strategy**	**30**	**90**	**180**	**30**	**90**	**180**	**30**	**90**	**180**
$P/E = 1.08$	HS_P	-0.192	-0.116	-0.048	1.008	1.184	1.362	1.635	1.709	2.018
	$HS_{(P,MSR^2)}$	-0.046	-0.048	-0.007	0.614	0.819	0.794	0.878	0.991	1.052
	$HS_{(P,MSR^2,MSR^3}$	0.038	-0.039	-0.006	0.665	0.735	0.788	0.863	0.984	1.035
	$HS_{(P,\emptyset)}$	-0.057	-0.049	-0.008	0.787	0.916	0.953	1.137	1.268	1.406
$P/E = 1.05$	HS_P	-0.221	-0.094	-0.037	1.206	1.354	1.515	1.876	2.027	2.205
	$HS_{(P,MSR^2)}$	-0.059	-0.021	-0.002	0.768	0.774	0.856	1.038	1.046	1.092
	$HS_{(P,MSR^2,MSR^3}$	0.057	-0.017	-0.001	0.735	0.769	0.812	1.012	1.028	1.034
	$HS_{(P,\emptyset)}$	-0.068	-0.028	0.003	1.003	1.109	1.294	1.337	1.429	1.501
$P/E = 1.025$	HS_P	-0.222	-0.088	-0.027	1.471	1.531	1.659	2.269	2.272	2.416
	$HS_{(P,MSR^2)}$	0.049	-0.015	-0.007	0.878	0.841	0.757	1.382	1.164	1.119
	$HS_{(P,MSR^2,MSR^3}$	0.045	-0.012	-0.006	0.872	0.833	0.746	1.358	1.118	1.104
	$HS_{(P,\emptyset)}$	-0.058	-0.017	-0.008	1.187	1.175	1.213	1.514	1.606	1.713

Notes: The values of the mean hedging error $\overline{\mathcal{HE}}$, the mean absolute error MAE and the root mean squared error RMSE are displayed in this Table for various hedging strategies executed on European options written on the S&P 500 index. $\mathrm{HS_P}$ is a delta hedging strategy on the underlying stock. $\mathrm{HS_{(P,MSR^2)}}$ is a two-instrument strategy, $\mathrm{HS_{(P,MSR^2,MSR^3)}}$ is a three-instrument strategy, and $\mathrm{HS_{(P,\emptyset)}}$ is the cross-hedging strategy. The third-order non-central risk-neutral moment at each point in time is used to approximate the third-order moment swap rate. The deltas of the hedging instruments are drawn from the estimates of the model in Table 3. Maturities considered are 30, 90 and 180 days. Hedging errors are computed each day with daily rebalancing from 1/2003 to12/2017.

the hedging strategies involving higher-order moment swaps perform better across all moneyness and maturity categories. Even though the moment swap rates in equations (4) and (5) are obtained by convergence in probability, there is no evidence of model misspecification and the results of the hedging strategies are robust. The mean hedging error moves from negative (delta hedged case) to positive or very close to zero because of better hedging and not because the new tools used over hedge. An examination of the left tail of the distribution of the hedging errors ($\mathcal{HE}$) of strategy HS_P confirms the superiority of strategies $HS_{(P,MSR^2,MSR^3)}$ and $HS_{(P,MSR^2,MSR_t^{12})}$, consistent with the theory that jump risk is priced by the market. Hence, strategies $HS_{(P,MSR^2,MSR^3)}$ and $HS_{(P,MSR^2,MSR_t^{12})}$ adequately captures the mean jump risk premium.

6. Conclusions

This paper provides the evidence for the importance of considering stochastic volatility, random jumps, and higher-order moment swaps in the pricing and hedging model. The model considered here assumes jumps to occur in the price process. A model with correlated jumps between the asset return and its volatility is being treated and tested in a separate paper. The model is tested using return data as well as European call and put option data on the S&P 500 index. Pricing accuracy is assessed by imposing consistency between physical and risk-neutral estimates. Using a two-step iterative approach, the paper first filters latent model variables and then uses these variables to estimate model parameters. These two steps are iterated until there is no further improvement in the aggregate objective function. An integrated approach is used by analyzing option data as well as the underlying return data in the calibration process. The results of the empirical tests show that adding jump components to a stochastic volatility model in a market enlarged with higher-moment swaps leads to a more realistic modeling of conditional higher moments as well as the moneyness and maturity effects, an improvement of the modeling of the term structure of the conditional variance, and a superior model pricing performance. Short-term option prices are critically influenced by the mean jump risk premium. The model fit to the time series data and cross-sectional dimension is outstanding, a confirmation of the model's ability to capture more of the variability of the option data. The root mean squared errors reported both in-sample and out-of-sample are much smaller than those reported under the two-factor stochastic volatility model. Moreover, the test results confirm that jump risk is priced by the market.

The performance of alternative hedging strategies is evaluated using European options and variance swaps data written on the S&P 500 index. The results show that to hedge against stochastic volatility and random jumps, the self-financing portfolio must contain variance and higher-order moment swaps. Under this condition, a perfect hedge of derivative securities can be achieved when the state variable follows the stochastic volatility with random jumps model in a market enlarged with higher-order moment swaps. The key new contribution of this paper is that hedging strategies, driven by the stochastic volatility with random jumps model in a market enlarged with higher-order moment swaps, perform better across all moneyness and maturity classes. The calibration methodology adopted leads to computed pricing and hedging errors that are much lower than those under the two-factor stochastic volatility model. This research shows the importance of considering stochastic volatility, random jumps, and higher-order moment swaps in the pricing and hedging model.

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