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Optimal Portfolios under a Conditional Value-at-Risk Constraint

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Abstract

This paper demonstrates how CVaR methodology can be applied to optimal portfolio problem and analyses the decision rules for consumption and asset allocation in the optimal portfolio. CVaR is able to quantify dangers beyond VaR and moreover it has superior properties in many respects. This provides a good way to control market risk in the continuous time portfolio selection. The problem is formulated as a constrained maximization of expected utility based on the power utility function. We used the analysis methods presented in K.F.C. Yiu (2004). The dynamic programming technique is applied to derive the HJB equation, the method of Lagrange-Kuhn-Tucker is used to tackle the constraint and numerical method is proposed to solve the HJB equation and the optimal constrained portfolio allocation. In this paper, we also compare the optimal portfolio depended on CVaR risk constraint which derived for three asset loss distributions. We find that investments in risky asset are even reduced under the imposed CVaR constraint which was measured by non-normally distributed assumption.

Keywords: Optimal portfolio; Value-at-risk; Conditional Value-at-Risk; Dynamic programming

1. Introduction

The fundamental concept behind optimal portfolio problem is how much wealth to allocate to current consumption and how much to save/invest for future consumption. The decision to allocate savings among the available investment opportunities is called portfolio selection. The basic portfolio selection problem was introduced by Markowitz (1952) in a static framework, taking the individual's consumption decision as given. Markowitz developed mean-variance analysis in the context of selecting a portfolio of common stocks which makes a one-off decision at the beginning of the period and holds on until the end of the period. Gradually, researchers have continued to extend the single-period model to continuous-time models. In the pioneering work of Merton (1971), the portfolio problem was reduced to a control problem which can be solved by applying results from stochastic dynamic programming. Explicit solutions for a particular class of utility functions have been obtained.

In the financial economic, VaR is a very popular used concept for quantifying the downside risk of portfolio. For a discussion of VaR as a risk measure see, for instance, Elton, Gruber, Brown, and Goetzmann (2003) or Jorion (1996). Klüppelberg and Korn (1998) and Alexander and Baptista (1999) conduct a Mean-VaR analysis using VAR to manage the risk of portfolios and compare the result to mean-variance approach. However, these studies are static framework. Luciano (1998), Basak and Shapiro (2001) and K.F.C. Yiu (2004) focus on optimal portfolio choice of a utility-maximizing agent and use the VaR as a constraint. In particular, Luciano (1998) have compared the deviations from static VaR measures to the dynamic ones rather than applying VaR constraint to optimal portfolio problem. Basak and Shapiro (2001) is the first attempt to directly embed risk management objectives into a utility-maximizing framework which focus on imposing the VaR constraint at one point in time to study trading between recalculated VaRs. K.F.C. Yiu (2004) looks at the continuous time optimal portfolio problem when a VaR constraint is imposed. The dynamic programming technique is applied and the method of Lagrange multiplier is used to solve the constrained portfolio allocation.

Unfortunately, VaR is not a coherent risk measure as discussed by Artzner et al. (1997, 1999). It has undesirable mathematical characteristics such as a lack of subadditivity and convexity. For example, VaR associated with a combination of two portfolios can be greater than the sum of the risks of the individual portfolios. Motivated by this shortcoming, Rockafellar and Uryasev (2000) introduced the CVaR risk measure for continuous distribution functions, a simple description of the convex optimization problems with CVaR constraints can be found in the paper. Recently, Pflug(2000) proved that CVaR is a coherent risk measure, see also Acerbi (2001), Acerbi and Tasche (2001). CVaR has many attractive properties including transition-equivariant, positively homogeneous, convex with respect to portfolio positions, monotonic with restrict to stochastic dominance. See Ogryczak and Ruszczynski (2002) for an overview of CVaR. In addition, minimizing CVaR typically leads to a portfolio with a small VaR. In this paper, we extend the VaR minimization approach, developed in K.F.C. Yiu (2004), to optimal portfolio problems with CVaR constraint. In the literature, researchers have been aware that the risk of extreme, rare events, such as the Black Monday 1987, cannot always be accurately described by the normally-distributed random variable. We show that this approach is also possible to extend to risk constraint which derived for different asset loss distributions. In this paper, we provide a constrained portfolio choice problem with the CVaR constraint and discuss the portfolio choice implications.

The rest of this article proceeds as follows. First, we introduce the risk measure of CVaR and derive it from different asset loss distribution assumption. Then, the optimal portfolio problem is formulated as a constrained maximization of the expected power function utility, with the constraint being the CVaR which is derived to model market risk for n risky assets plus a risk-free asset. Dynamic programming is applied to solving the Hamilton–Jacobi–Bellman equation (HJB-equation) coupled with the CVaR constraint, and the method of Lagrange-Kuhn-Tucker is then applied to handle the constraint. Then, a numerical method is proposed to solve the HJB-equation and hence the constrained optimal portfolios. Finally, the result with CVaR constraint is compared.

2. Conditional Value-at-Risk

In the financial industry, VaR is a very popular measure of risk for quantifying the downside risk of portfolio. The definition of VaR is the maximum expected loss in a specified horizon period at a given confidence level. In addition, CVaR is the loss one expects to suffer at that confidence level by holding it over the investment period, given that the loss is equal to or larger than its VaR. This risk measure does account for the loss size concerning events when the loss exceeds VaR. It has been shown to satisfy the requirements of the coherent risk measures and is consistent with the second degree stochastic dominance.

Assume that portfolio's rates of return have a multivariate normal distribution is a popular assumption in the literature. Let $\Phi(\bullet)$ be the standard normal cumulative distribution function and $\phi(\bullet)$

the standard normal probability density function. Define $E(r_\omega)$ as the portfolio ω's random rate of return, and let $\sigma(r_\omega)$, $f(\bullet)$ and $F(\bullet)$ denote, respectively, its expected rate of return, standard deviation, probability density function and the cumulative distribution function of . Given an investment period and a confidence level ζ (e.g., ζ = 0.99), using the definition of VaR, we have that

$$VaR(r_\omega,\zeta) = -F_{r_\omega}^{-1}(\zeta) = -E(r_\omega) - z_\zeta \sigma(r_\omega)\,{}^{1} \tag{1}$$

Similarly, it follows from the definition of CVaR that

$$CVaR(r_\omega,\zeta) = -E\{r_\omega \mid r_\omega \le -VaR(r_\omega,\zeta)\} = -E(r_\omega) - k_\zeta \sigma(r_\omega), \tag{2}$$

where

$$k_\zeta = \frac{\int_{-\infty}^{z_\zeta} x\phi(x)dx}{1-\zeta}. \tag{3}$$

Equation (2) implies that $k_\zeta > z_\zeta$ at the given confidence level ζ, thus we have $CVaR(r_\omega,\zeta) > VaR(r_\omega,\zeta)$. From equation (1) and (2), CVaR evaluates the risk of an investment in a conservative way.

In actually, calculation of VaR or CVaR based on the normal probability measure may underestimate the exposed risk. Therefore, we also consider the *t*-distribution and Extreme Value distribution for CVaR calculation. $CVaR_{normal}$ is shown as equation (2). The *t*-distribution[2] is symmetric and bell-shaped, like the normal distribution, but has heavier tails which means that it is more prone to producing values that fall far from its mean. We can get $CVaR_{t\text{-}distribution}$ by directly substituting the probability density function and the cumulative distribution function into equation (2). The Extreme Value distribution is designed as a mixture distribution which take the behavior of extreme events and non-symmetric into account. It is made by adding a catastrophic loss event with probability p to the normal distribution. Let B(L,p) be the Bernoulli distribution which take value $L = \Phi^{-1}(10^{-7})$ with success probability p = 0.3. $CVaR_{extreme}$ is formulated as

$$CVaR_{extreme}(r_\omega,\zeta) = -E(r_\omega) - \left[\frac{\int_{-\infty}^{z_\zeta} x\cdot\phi(x)dx}{1-\zeta} + pL\right]\sigma(r_\omega). \tag{4}$$

3. Continuous-Time Optimal Portfolios

The optimal problem, based on the framework by Merton, is formulated as maximizing the total expected utility by allocating personal wealth among current consumption, investment in a riskless asset, and investment in n risky assets. Assume at time t = 0, the agent is endowed with initial wealth X_0 and, in the financial market, the agent can invest money in a risk-free asset *B* at the deterministic short rate of interest *r*. The change in riskless asset can be written as

$$dB(t) = rB(t)dt. \tag{5}$$

Alternatively, the agent can also invest in n risky assets with the price process as $(S_1(t),\dots,S_n(t))$, where the vector process *S(t)* follow the standard Wiener process

$$dS(t) = D(S(t))\mu dt + D(S(t))\sigma dW(t). \tag{6}$$

W(t) is a k-dimensional standard Wiener process, μ is an n-vector, σ is an n×k matrix, and *D(S(t))* is the diagonal matrix diag$[S_1(t),\dots,S_n(t)]$. Let $\omega(t)$, an n-vector, be the amount of wealth

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[1] $z_\zeta = \Phi^{-1}(1-\zeta)$

[2] The probability density function of t-distribution is $f(x) = \dfrac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi}\cdot\Gamma(\frac{v}{2})}\cdot(1+\frac{x^2}{v})^{-\frac{(v+1)}{2}}$.

invested in *S(t)* and define ***e*** to be the n-vector of 1, then the budget-constraint dynamic in a portfolio consisting of *B(t)* and *S(t)* with consumption *c(t)* is therefore

$$\begin{aligned} dX(t) &= \frac{X(t) - \omega(t)'\boldsymbol{e}}{B_t} dB_t + \omega(t)' \times D(S(t))^{-1} dS(t) - c(t)dt \\ &= (X(t) - \omega(t)'\boldsymbol{e})rdt + \omega(t)'(\mu dt + \sigma dW(t)) - c(t)dt \\ &= \left(\omega(t)'(\mu - r\boldsymbol{e}) + rX(t) - c(t)\right)dt + \omega(t)'\sigma dW(t) \end{aligned} \quad (7)$$

The deterministic portion is composed of the return on the funds in the sure asset, plus the expected return on the funds in the risky asset, less consumption. We specified that the utility function is a power function of the form

$$U(c(x,t),t) = e^{-\delta t} c(x,t)^{\gamma} \quad (8)$$

where $\delta > 0$ and $0 < \gamma < 1$. The economic reasoning behind this is that we now have an infinite marginal utility at $c = 0$. This will force the optimal consumption plan be positive throughout the planning period. Consider the unconstrained optimal portfolio problem where agent needs continuous consumption over the given period of time. Then, the objective system is maximization of the expected utility stream

$$\underset{\omega(t),c(t)}{Max} \; E\left[\int_0^T e^{-\delta t} c(x,t)^{\gamma} dt\right], \quad (9)$$

subject to

$$dX(t) = \left(\omega(t)'(\mu - re) + rX(t) - c(t)\right)dt + \omega(t)'\sigma dW(t) \quad (10)$$

Merton (1971) has derived the analytical solution for utility function of this form.

4. Dynamic CVaR Constraint

In this section, the continuous-time model presented in section 3 can be extended to a constrained portfolio choice problem with CVaR constraint. Similarly, using the analysis method in K.F.C Yiu (2004), we can derive the CVaR constraint in continuous time. First, the budget constraint dynamic is rewritten as

$$dX(t) = \alpha(\theta(t) - X(t))dt + \omega(t)'\sigma dW(t), \quad (11)$$

where $\alpha = -r$ and $\theta(t) = \dfrac{\omega(t)'(\mu - re) - c(t)}{-r}$. Let $\Delta t = (s - t)$ be the time horizon period, then, integrating both side of equation (11) and rearranging the mathematical results, we have

$$X(s) = e^{-\alpha(s-t)}(X(t) - \theta(t)) + \theta(t) + \int_t^s e^{-\alpha(s-\tau)}\omega(t)'\sigma dW(\tau) \quad (12)$$

Process such as equation (12) is called Ornstein–Uhlenbeck process except that the speed-of-adjustment parameter α is negative instead. Define the loss by $\Delta X(t) = X(s) - e^{r(s-t)}X(t)$ to eliminate *X(t)* from equation (12). Using the conditional mean and conditional variance[3] of the process on time t, the CVaR constraint by its definition of equation (2) is given by

$$CVaR_t = -(\theta(t) - \theta(t)e^{r\Delta t}) - \left(\int_{-\infty}^{x_{1-\zeta}} \frac{f(x)}{F(x_{1-\zeta})} dx\right) \times \sqrt{\frac{e^{2r\Delta t}}{2r}\omega(t)'\Sigma\omega(t)} \quad (13)$$

[3] The conditional mean and conditional variance of *X(s)* is given by $E_t[X(s)] = \theta(t) + e^{-\alpha(s-t)}(X(t) - \theta(t))$ and $V_t[X(s)] = \dfrac{\omega(t)'\Sigma\omega(t)}{2\alpha}(1 - e^{-2\alpha(s-t)})$, where $\Sigma = \sigma\sigma'$.

Substituting $\theta(t)$ into equation (13), the constraint of restricting the CVaR at level R is presented as

$$a_1\sqrt{\omega(t)'\Sigma\omega(t)} + a_2\omega(t) + bc(t) \le R \quad (14)$$

where the coefficients in equation (14) is

$$a_1 = \left(-\int_{-\infty}^{x_{1-\zeta}} \frac{f(x)}{F(x_{1-\zeta})}dx\right)\times\sqrt{\frac{e^{2r\Delta t}}{2r}},\quad a_2 = -\frac{\mu - re}{r}(e^{r\Delta t}-1),\quad b = \frac{1}{r}(e^{r\Delta t}-1) \quad (15)$$

Equation (15), in fact, imposes an upper bound on $\omega(t)$ to constrain the investment in the risky asset. The final optimal portfolio problem with CVaR constraint is given by

$$\underset{\omega(t),c(t)}{Max}\ E\left[\int_0^T e^{-\delta t}c(x,t)^{\gamma}dt\right], \quad (16)$$

subject to

$$\begin{aligned} &dX(t) = (\omega(t)'(\mu - re) + rX(t) - c(t))dt + \omega(t)'\sigma dW(t) \\ &a_1\sqrt{\omega(t)'\Sigma\omega(t)} + a_2'\omega(t) + bc(t) \le R \end{aligned} \quad (17)$$

5. Optimality Conditions and Numerical Methods

In this section, we present the general solution method for the optimal portfolio problem. The dynamic programming methodology developed by Bellman is applied. The optimal portfolio problem is shown to be equivalent to the problem of finding a solution to the HJB-equation (Bjork, 1998; K.F.C. Yiu, 2004). To derive the optimality equations, we restate equation (15) in a dynamic programming form so that the Bellman principle of optimality can by applied. To do this, define

$$J(x,t) = \underset{\omega(t),c(t)}{Sup}\ E\left[\int_t^T U(c(t),t)dt\right], \quad (18)$$

where *x* is a possible state of *X(t)*. Denote

$$G(x,\omega(x,t),c(x,t)) \equiv \omega(x,t)(\mu - re) + rx - c(x,t)$$

and

$$H(\omega(x,t)) \equiv \omega(x,t)'\Sigma\omega(x,t),$$

the corresponding HJB-equation is given by

$$\begin{aligned} \frac{\partial J}{\partial t} + \underset{\omega(x,t),c(x,t)}{Sup}&\left(U(c(x,t),t) + G(x,\omega(x,t),c(x,t))\frac{\partial J}{\partial x}\right. \\ &\left. + \frac{1}{2}H(w(x,t))\frac{\partial^2 J}{\partial x^2}\right) = 0 \end{aligned} \quad (19)$$

To complete the solution, the boundary conditions need to be applied as

$$J(0,t) = 0,\quad J(x,T) = 0. \quad (20)$$

The static optimization problem to be solve by the Lagrange-Kuhn-Tucker methods for constrained optimization is

$$\underset{\omega(x,t),c(x,t)}{max}\left(U(c(x,t),t) + G(x,\omega(x,t),c(x,t))\frac{\partial J}{\partial x} + \frac{1}{2}H(\omega(x,t))\frac{\partial^2 J}{\partial x^2}\right) \quad (21)$$

subject to the constraint

$$a_1\sqrt{H(\omega(t))} + a_2'\omega(x,t) + bc(x,t) \le R. \quad (22)$$

To provide a solution, denote the Lagrange function as

$$\Im(\omega(x,t),c(x,t),\lambda(x,t))=$$
$$U(c(x,t),t)+G(x,\omega(x,t)),c(x,t))\frac{\partial J}{\partial x}+\frac{1}{2}H(\omega(x,t))\frac{\partial^2 J}{\partial x^2} \tag{23}$$
$$-\lambda(x,t)\left(R-a_1\sqrt{H(\omega(x,t))}+a_2'\omega(t)+bc(t)\right),$$

where $\lambda(x,t) \leq 0$ is the multiplier. Then using the extreme points from the first-order necessary conditions[4] of the static optimization problem, we can get $\omega_{opt}(x,t)$, $c_{opt}(x,t)$ and $\lambda_{opt}(x,t)$. When equation (19) is evaluated at the optimum t $\omega_{opt}(x,t)$, $c_{opt}(x,t)$ and $\lambda_{opt}(x,t)$., the optimization problem is simplified as

$$\frac{\partial J}{\partial t}+U(c_{opt}(x,t),t)+G(x,\omega_{opt}(x,t),c_{opt}(x,t))\frac{\partial J}{\partial x}+\frac{1}{2}H(\omega_{opt}(x,t))\frac{\partial^2 J}{\partial x^2}=0 \tag{24}$$

It becomes an ordinary differential equation which can be solved for the optimal value fuction $J_{opt}(x,t)$. The first-order conditions together with numerical methods are required to solve for$\omega_{opt}(x,t)$, $c_{opt}(x,t)$, $\lambda_{opt}(x,t)$ and $J_{opt}(x,t)$ iteratively.

Merton (1971) has derived the analytical solution for the value function $J(x,t)$ for utility function of this form. He suggest a trial value function of the form

$$J(x,t)=e^{-\delta t}h(t)x^{\gamma}, \tag{25}$$

which separates the x and t variables.[5] Substituting the trial function into the HJB equation, it reduces to a Bernoulli equation for $h(t)$ which is an ordinary differential equation. Substituting the utility function into the equation (24), the HJB equation is given by

$$\frac{\partial J}{\partial t}+e^{-\delta t}c_{opt}^{\gamma}(x,t)+\left(\omega_{opt}(x,t)(\mu-r)+rx-c_{opt}(x,t)\right)\frac{\partial J}{\partial x}$$
$$+\frac{1}{2}\omega_{opt}^2(x,t)\sigma^2\frac{\partial^2 J}{\partial x^2}=0 \tag{26}$$

(26) Finally, substitute for the derivative in equation (26), dividing by $e^{-\delta t}x^{\gamma}$ the problem is transformed into

$$h_t(x,t)+A(\omega_{opt}(x,t),x)h(x,t)+B(c_{opt}^{\gamma}(x,t),h(x,t))=0, \tag{27}$$

with the terminal condition

$$h(x,T)=0, \tag{28}$$

where

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[4] The first-order condition are given by
$$\frac{\partial\Im}{\partial w}=0\Rightarrow(\mu-re)\frac{\partial J}{\partial x}+\Sigma\omega(x,t)\frac{\partial^2 J}{\partial x^2}+\lambda(x,t)(a_1\frac{\Sigma\omega(x,t)}{\sqrt{H(\omega(x,t))}}+a_2)=0$$
$$\frac{\partial\Im}{\partial c}=0\Rightarrow\frac{\partial U}{\partial c}-\frac{\partial J}{\partial x}+\lambda(x,t)b=0$$
$$\frac{\partial\Im}{\partial\lambda}\cdot\lambda(x,t)=0\Rightarrow\lambda(x,t)\left(R-a_1\sqrt{H(\omega(x,t))}+a_2'\omega(t)+bc(t)\right)=0$$
$$\lambda(x,t)\leq 0$$

[5] The derivative of x, x^2 and t is given by $J_x=\gamma e^{-\delta t}h(t)x^{\gamma-1}$, $J_{xx}=\gamma(\gamma-1)e^{-\delta t}h(t)x^{\gamma-2}$ and $J_t=e^{-\delta t}h_t(t)x^{\gamma}-\delta e^{-\delta t}h(t)x^{\gamma}$.

$$A(\omega_{opt}(x,t),x)=\gamma\left(\frac{\omega_{opt}(x,t)(\mu-r)}{x}+r\right)+\frac{1}{2}\frac{\omega_{opt}^{2}(x,t)\sigma^{2}\gamma(\gamma-1)}{x^{2}}-\delta$$
$$B(c_{opt}^{\gamma}(x,t),h(x,t))=\frac{c_{opt}^{\gamma}(x,t)}{x^{\gamma}}-\frac{\gamma h(t)c_{opt}(x,t)}{x} \quad (29)$$

To avoid the singularity in calculating negative powers of $h(x,t)$ near to the terminal time T, equation (26) is transformed into[6]

$$g_t(x,t)+(1-\beta)A(\omega_{opt}(x,t),x)g(x,t)+(1-\beta)B(c_{opt}(x,t),g(x,t))g(x,t)^{\gamma}=0 \quad (30)$$

where

$$g(x,T)=0 \quad (31)$$

To evaluate these expressions numerically we use an iterative algorithm procedure to find an optimal solution. We take an initial guess, the unconstrained solution $\omega=\frac{(\mu-r_f)}{\sigma^2(1-\gamma)}x$, to solve for the optimal portfolio problem. Dividing the computational domain into a grid of $N_x \times N_t$.We repeat the following procedure until convergence:

Step 1. Set $k=0$. $\lambda_{opt}^{(0)}=0$, $\omega_{opt}^{(0)}=\frac{(\mu-r_f)}{\sigma^2(1-\gamma)}x$, $c_{opt}^{(0)}=x\left(h(t)^{(0)}\right)^{-1/1-\gamma}$ and calculate the derivative of x, $J_x^{(0)}$, and the derivative of x^2, $J_{xx}^{(0)}$. Then solve the unconstrained solution $g(t)_{N_t-1}^{(0)} \sim g(t)_0^{(0)}$.

Step 2. For $x=[0,\Delta x,...,N_x\Delta x]$ and $t=[(N_t-1)\Delta t,...,\Delta t,0]$, calculate $\lambda_{opt}^{(1)}$ and $c_{opt}^{(1)}$ from the first-order condition of the Lagrange function

$$\lambda_{opt}^{(k+1)}\left(R-(a_1\sigma+a_2)\omega_{opt}^{(k+1)}-bc_{opt}^{(0)}\right)=0$$
$$\omega_{opt}^{(k+1)}=\frac{-J_x^{(0)}(\mu-r_f)-\lambda_{opt}^{(k+1)}(a_1\sigma+a_2)}{\sigma^2 J_{xx}^{(0)}} \quad (32)$$
$$\gamma\left(c_{opt}^{(k+1)}\right)^{\gamma-1}=\gamma x^{\gamma-1}h(t)^{(0)}+\lambda_{opt}^{(k+1)}b=0$$

Step 3. For $x=[0,\Delta x,...,N_x\Delta x]$ and $n=N_t-1,...,0$, solve

$$g(t)_n^{(k+1)}=g(t)_{n+1}^{(k+1)}+\Delta t(1-\beta)A(\omega_{opt}^{(k+1)},x)g(t)_{n+1}^{(k+1)}$$
$$+\Delta t(1-\beta)B(c_{opt}^{(k+1)},g(t)_n^{(k)})\left(g(t)_n^{(k)}\right)^{\gamma}, \quad (33)$$

with the boundary condition, $g(t)_{n+1}^{(k+1)}=g(t)_T^{(k+1)}=0$

Step 4. Return to Step 2 with k=k+1 until convergence requirement is fulfilled.

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[6] $g(t)=h(t)^{1-\beta}$, $\beta=-\frac{\gamma}{1-\gamma}$

6. Numerical Results and Conclusion

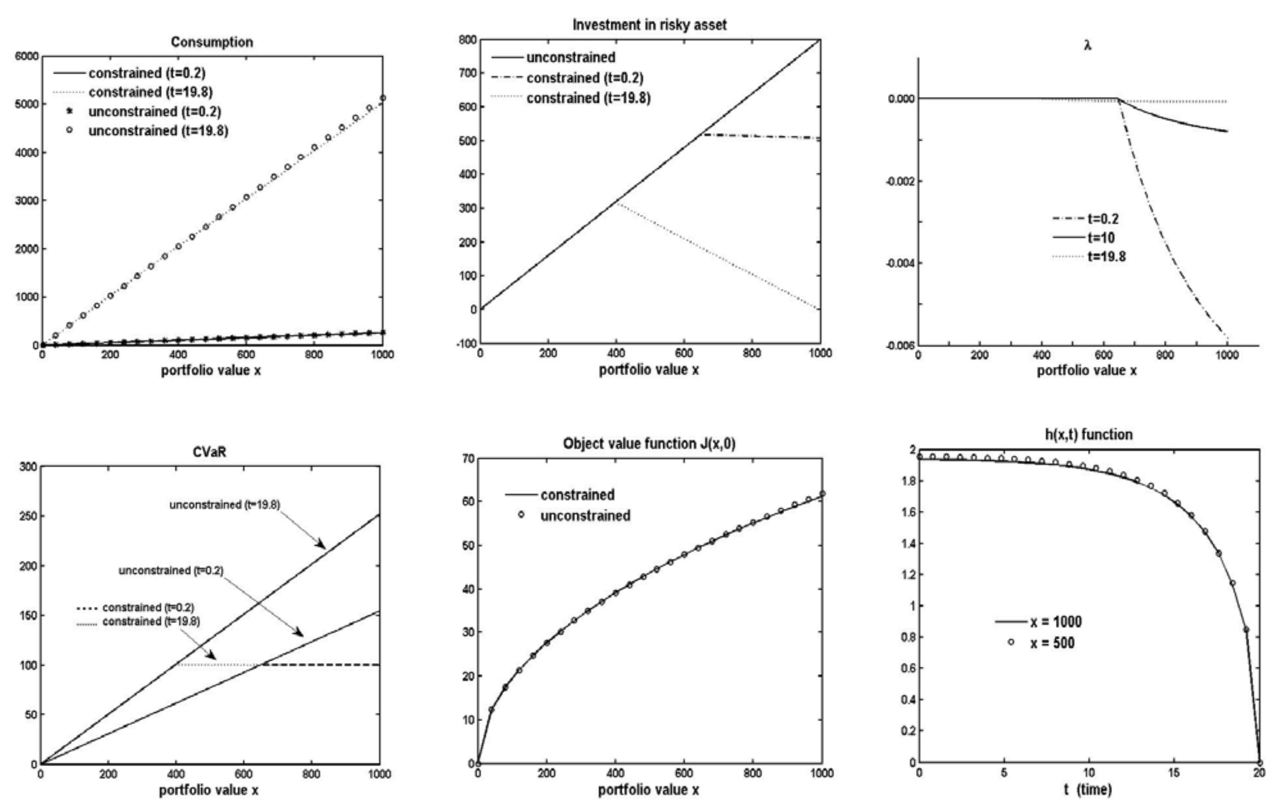
The optimal portfolio choice and consumption rule given in equation (16) and (17) is done using a Matlab program. The terminal year is chosen to be 20 and N_t is fixed at 1000. Thus, the corresponding horizon period is Δt =1/50. The wealth has the range between 0 to 1000, where $\Delta x = 2$ and $N_x = 500$. We consider investors with coefficients of relative risk aversion, γ, equal to 0.3 or 0.5 and a comparison of the coefficients of dynamic process is given, see Table 1. The motivation for these economic parameters comes from the common financial advice that agent with slightly risk aversion will place more percentage wealth to risky asset. Also, if the expected growth rate of wealth level is larger and the uncertainty of wealth is less, they will invest more money in the risky asset.

Table 1: Parameter assumptions in continuous-time model

Parameter	Asset dynamic process	Utility function
Case A	$\mu = 0.2, \sigma = 0.5, r = 0.1$	$\delta = 0.2, \gamma = 0.5$
Case B	$\mu = 0.12, \sigma = 0.2, r = 0.05$	$\delta = 0.1, \gamma = 0.3$
Case C	$\mu = 0.12, \sigma = 0.2, r = 0.05$	$\delta = 0.1, \gamma = 0.5$

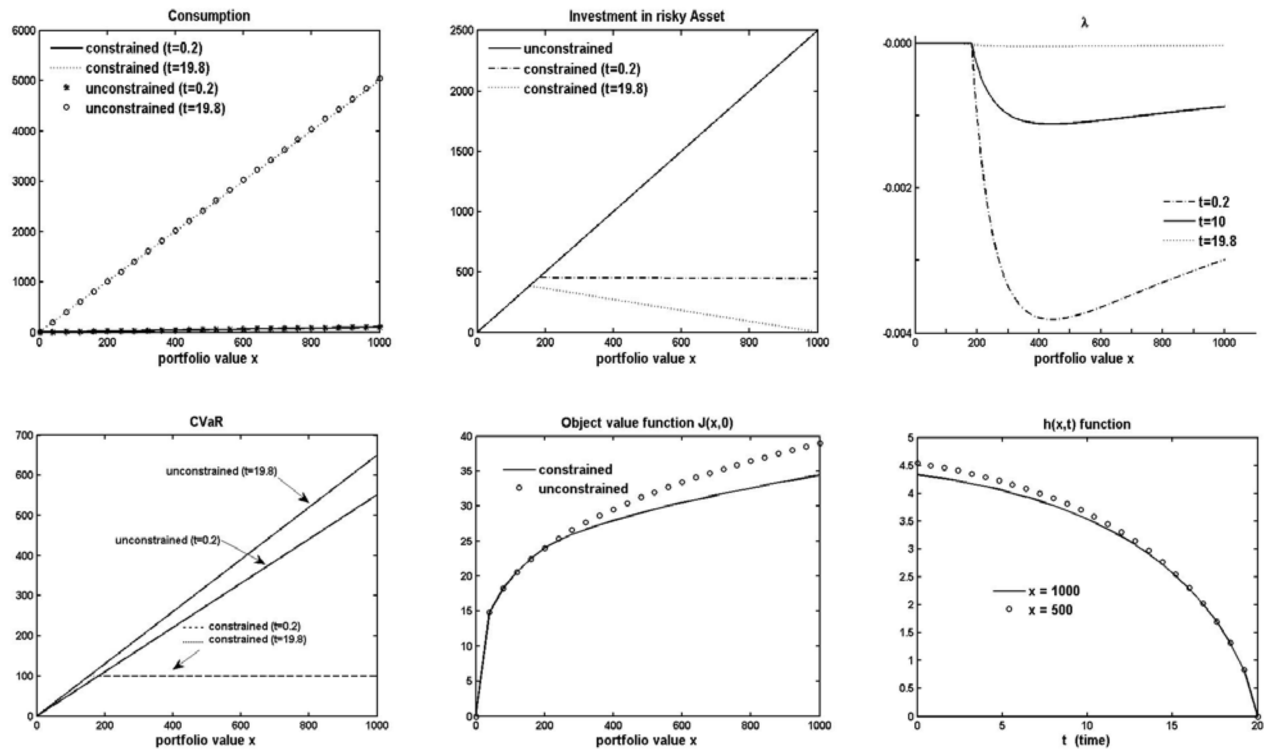
Figure 1 through Figure 3 plot the result of comparative exercises that the optimal portfolio rule has been solved with CVaR constraint (The detail numerical results of consumption, investment in risky asset and object value function are showed in Table 2 to Table 4). From the figure, we can observe that well control has been achieved and the allocation to the risky asset have been reduced in order to fulfil the dynamic CVaR constraint. In addition, when the constraint is not active, Lagrange

Figure 1: Optimal portfolio based on Normal-distributed CVaR constraint



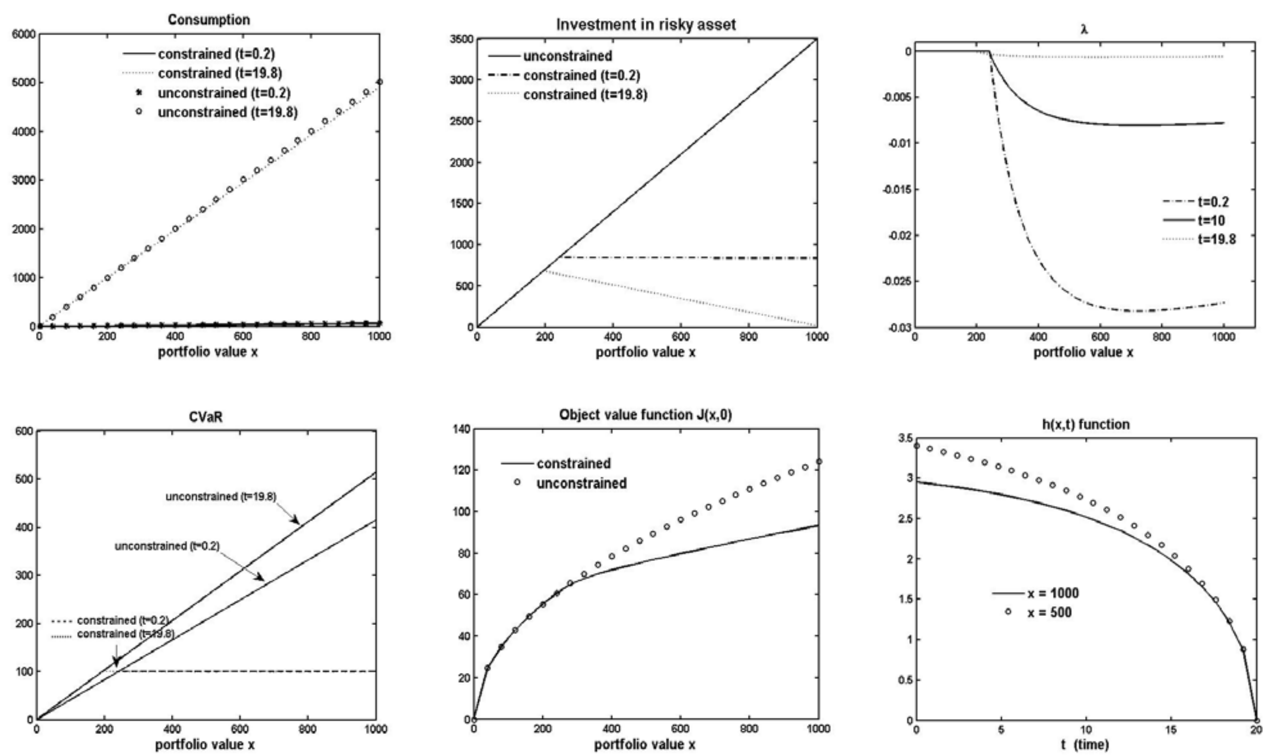
In this case, the parameters are $\mu = 0.2$, $\sigma = 0.5$, $r = 0.1$, $\delta = 0.2$, $\gamma = 0.5$, the horizon period is Δt =1/50, the maximum loss (CVaR) allowed is R=100 with probability ζ=0.01.

Figure 2: Optimal portfolio based on T-distributed CVaR constraint



In this case, the parameters are μ = 0.12, σ = 0.2, r = 0.05, δ = 0.1, γ = 0.3, the horizon period is Δt =1/50, the maximum loss (CVaR) allowed is R=100 with probability ζ=0.01.

Figure 3: Optimal portfolio based on Extreme-Value-distributed CVaR constraint



In this case, the parameters are μ = 0.12, σ = 0.2, r = 0.05, δ = 0.2, γ = 0.5, the horizon period is Δt =1/50, the maximum loss (CVaR) allowed is R=100 with probability ζ=0.01.

Table 2: Consumption pattern (C_t) for different portfolio value at time t = 0.2 and t = 19.8

CVaR Constraint Based on	**Portfolio value X_0**									
	100	**200**	**300**	**400**	**500**	**600**	**700**	**800**	**900**	**1000**
t = 0.2	Case A: $\mu = 0.2, \sigma = 0.5, r = 0.1; \delta = 0.2, \gamma = 0.5$									
Unconstrained	26.15	52.30	78.46	104.61	130.76	156.91	183.06	209.22	235.37	261.52
Normal	26.15	52.30	78.46	104.61	130.76	156.91	182.76	208.33	233.91	259.48
T-distribution	26.15	52.30	78.31	104.27	130.23	156.19	182.15	208.11	234.07	260.03
Extreme Value	26.15	52.30	78.46	104.61	130.44	156.24	182.03	207.82	233.62	259.41
	Case B: $\mu = 0.12, \sigma = 0.2, r = 0.05; \delta = 0.1, \gamma = 0.3$									
Unconstrained	10.36	20.72	31.08	41.44	51.79	62.15	72.51	82.87	93.23	103.59
Normal	10.36	20.72	31.08	41.44	51.79	61.97	72.05	82.13	92.21	102.29
T-distribution	10.36	20.70	30.96	41.23	51.49	61.76	72.02	82.29	92.55	102.82
Extreme Value	10.36	20.72	31.08	41.32	51.51	61.69	71.88	82.06	92.24	102.43
	Case C: $\mu = 0.12, \sigma = 0.2, r = 0.05; \delta = 0.1, \gamma = 0.5$									
Unconstrained	6.55	13.10	19.65	26.20	32.75	39.30	45.85	52.40	58.95	65.50
Normal	6.55	13.10	19.65	26.16	32.46	38.77	45.07	51.38	57.68	63.99
T-distribution	6.55	13.04	19.51	25.97	32.44	38.91	45.37	51.84	58.31	64.77
Extreme Value	6.55	13.10	19.56	25.95	32.35	38.74	45.14	51.54	57.93	64.33
t = 19.8	Case A: $\mu = 0.2, \sigma = 0.5, r = 0.1; \delta = 0.2, \gamma = 0.5$									
Unconstrained	513.11	1026.23	1539.34	2052.45	2565.56	3078.68	3591.79	4104.90	4618.01	5131.13
Normal	513.11	1026.23	1539.34	2051.86	2546.76	3041.79	3536.89	4032.04	4527.21	5022.41
T-distribution	513.11	1025.49	1534.03	2042.57	2551.12	3059.67	3568.22	4076.77	4585.32	5093.87
Extreme Value	513.11	1026.23	1538.73	2042.03	2545.37	3048.74	3552.11	4055.50	4558.88	5062.27
	Case B: $\mu = 0.12, \sigma = 0.2, r = 0.05; \delta = 0.1, \gamma = 0.3$									
Unconstrained	504.21	1008.42	1512.62	2016.83	2521.04	3025.25	3529.46	4033.67	4537.87	5042.08
Normal	504.21	1008.42	1512.62	2006.44	2490.17	2974.04	3457.99	3941.99	4426.02	4910.08
T-distribution	504.21	1005.92	1504.73	2003.55	2502.37	3001.20	3500.02	3998.85	4497.68	4996.51
Extreme Value	504.21	1008.42	1507.21	2000.13	2493.10	2986.09	3479.09	3972.10	4465.11	4958.13
	Case C: $\mu = 0.12, \sigma = 0.2, r = 0.05; \delta = 0.1, \gamma = 0.5$									
Unconstrained	501.38	1002.75	1504.13	2005.51	2506.88	3008.26	3509.63	4011.01	4512.39	5013.76
Normal	501.38	1002.75	1498.50	1974.28	2450.31	2926.47	3402.69	3878.95	4355.25	4831.56
T-distribution	501.38	996.65	1490.87	1985.10	2479.33	2973.56	3467.80	3962.03	4456.27	4950.51
Extreme Value	501.38	1001.96	1488.77	1975.67	2462.62	2949.59	3436.56	3923.55	4410.54	4897.53

Multiplier is zero, the optimal portfolio follows the unconstrained solution. The negative Lagrangian multiplier, λ, indicate that the CVaR constraint has bound and the absolute size of the multiplier indicates how important it is associated. In the figure, kinks are produced whenever the CVaR constraint becomes active, this will make less allocation to the risky asset. On these figures, it produces very similar consumption patterns for each CVaR optimization. But, consumption has been affected but not greatly by the risk control. The reduction in the optimal object value function, $J_{opt}(x,0)$ is observed by imposing the CVaR constraint in these examples.

Table 2 to Table 4 summarizes the numerical results of the optimal consumption plan, investment rule and objective value function for all possible sceneries in the optimal portfolio model. Panel "Case A" of each table reports the optimal portfolio rule for a more conservative agent when financial market is more volatile. In contrast, Panel "Case B" of each table reports the result of a slightly risk averse agent subject to less uncertainty financial environment. Panel "Case C" of each table reports the result which designed for compared purpose. In Table 2, we summarize the estimating consumption path along x at time t = 0.2 and t = 19.8. Each Panel reveals the similar optimal consumption rule for the unconstrained and constrained CVaR results. For all parameter settings, the consumption rule has been affected but not greatly by the risk control. In Table 3, we explore the implication for portfolio choice of the risky asset at two different times. It shows that the optimal portfolio follows the unconstrained solution when the exposed risk has not yet exceeded the toleration level, R. By contrast, the proportions invested in the risky asset have been reduced in order to satisfy the risk management requirement. Because of the constant risk toleration level, the CVaR constraint becomes active, the allocation to the risky asset has decreased as the portfolio value x increases.

Table 3: Investment in risky asset (ω_t) for different portfolio value at time t = 0.2 and t = 19.8

CVaR Constraint Based on	Portfolio value X_0									
	100	200	300	400	500	600	700	800	900	1000
t = 0.2	Case A: $\mu = 0.2, \sigma = 0.5, r = 0.1; \delta = 0.2, \gamma = 0.5$									
Unconstrained	80.00	160.00	240.00	320.00	400.00	480.00	560.00	640.00	720.00	800.00
Normal	80.00	160.00	240.00	320.00	400.00	480.00	516.17	513.43	510.69	507.94
T-distribution	80.00	160.00	178.66	177.72	176.77	175.83	174.89	173.94	173.00	172.06
Extreme Value	80.00	160.00	240.00	320.00	327.85	326.11	324.37	322.64	320.90	319.16
	Case B: $\mu = 0.12, \sigma = 0.2, r = 0.05; \delta = 0.1, \gamma = 0.3$									
Unconstrained	250.00	500.00	750.00	1000.00	1250.00	1500.00	1750.00	2000.00	2250.00	2500.00
Normal	250.00	500.00	750.00	1000.00	1250.00	1334.22	1331.50	1328.77	1326.05	1323.32
T-distribution	250.00	453.34	452.41	451.47	450.54	449.60	448.67	447.73	446.80	445.86
Extreme Value	250.00	500.00	750.00	839.31	837.58	835.86	834.13	832.41	830.68	828.96
	Case C: $\mu = 0.12, \sigma = 0.2, r = 0.05; \delta = 0.1, \gamma = 0.5$									
Unconstrained	350.00	700.00	1050.00	1400.00	1750.00	2100.00	2450.00	2800.00	3150.00	3500.00
Normal	350.00	700.00	1050.00	1343.90	1342.20	1340.49	1338.79	1337.09	1335.38	1333.68
T-distribution	350.00	454.04	453.45	452.86	452.27	451.68	451.10	450.51	449.92	449.33
Extreme Value	350.00	700.00	842.99	841.91	840.83	839.74	838.66	837.58	836.50	835.41
t = 19.8	Case A: $\mu = 0.2, \sigma = 0.5, r = 0.1; \delta = 0.2, \gamma = 0.5$									
Unconstrained	80.00	160.00	240.00	320.00	400.00	480.00	560.00	640.00	720.00	800.00
Normal	80.00	160.00	240.00	315.69	262.60	209.51	156.40	103.29	50.18	-2.94
T-distribution	80.00	144.24	125.76	107.28	88.80	70.32	51.85	33.37	14.89	-3.59
Extreme Value	80.00	160.00	232.94	199.02	165.09	131.17	97.24	63.32	29.39	-4.53
	Case B: $\mu = 0.12, \sigma = 0.2, r = 0.05; \delta = 0.1, \gamma = 0.3$									
Unconstrained	250.00	500.00	750.00	1000.00	1250.00	1500.00	1750.00	2000.00	2250.00	2500.00
Normal	250.00	500.00	750.00	808.57	677.81	547.00	416.18	285.34	154.49	23.63
T-distribution	250.00	363.60	318.16	272.72	227.28	181.85	136.41	90.97	45.53	0.09
Extreme Value	250.00	500.00	591.07	507.59	424.11	340.62	257.14	173.65	90.16	6.67
	Case C: $\mu = 0.12, \sigma = 0.2, r = 0.05; \delta = 0.1, \gamma = 0.5$									
Unconstrained	350.00	700.00	1050.00	1400.00	1750.00	2100.00	2450.00	2800.00	3150.00	3500.00
Normal	350.00	700.00	945.88	817.27	688.58	559.86	431.13	302.38	173.62	44.86
T-distribution	350.00	364.44	319.42	274.40	229.38	184.36	139.34	94.32	49.30	4.28
Extreme Value	350.00	676.63	594.19	511.73	429.27	346.81	264.34	181.87	99.40	16.93

This is particularly so when t is close to the maturity time T. In addition, the allocations to risky asset become negative in some results. The agent behaves conservative toward the final stage and is willing to short sell risky asset in exchange for risk-free asset. We also find that investments in risky asset are reduced under the imposed CVaR constraint measured by non-normally distributed assumption. In Table 4, we show the object value function, $J_{opt}(x,0)$ for different portfolio value. With the imposed CVaR constraint, the object value function is smaller than that without the constraint.

Table 4: Object value function, $J_{opt}(x,0)$, for different portfolio value

CVaR Constraint Based on	Portfolio value X_0									
	100	200	300	400	500	600	700	800	900	1000
	Case A: $\mu = 0.2, \sigma = 0.5, r = 0.1; \delta = 0.2, \gamma = 0.5$									
Unconstrained	19.5574	27.6584	33.8745	39.1149	43.7318	47.9058	51.7441	55.3168	58.6723	61.8461
Normal	19.5582	27.6594	33.8757	39.1163	43.7334	47.9075	51.7218	55.1565	58.3051	61.2382
T-distribution	19.5582	27.6594	33.7116	38.5500	42.7472	46.5272	50.0026	53.2409	56.2867	59.1719
Extreme Value	19.5582	27.6594	33.8757	39.1163	43.6271	47.5426	51.0726	54.3261	57.3671	60.2370

Table 4: Object value function, $J_{opt}(x,0)$, for different portfolio value - continued

	Case B: $\mu = 0.12$, $\sigma = 0.2$, $r = 0.05$; $\delta = 0.1$, $\gamma = 0.3$									
Unconstrained	19.5192	24.0309	27.1392	29.5855	31.6338	33.4123	34.9937	36.4240	37.7341	38.9458
Normal	19.5236	24.0363	27.1453	29.5921	31.6406	33.3379	34.6121	35.6453	36.5365	37.3361
T-distribution	19.5236	23.9992	26.3484	27.9640	29.3133	30.5101	31.5992	32.6045	33.5413	34.4205
Extreme Value	19.5236	24.0363	27.1453	29.4456	30.9874	32.1942	33.2314	34.1650	35.0264	35.8324
	Case C: $\mu = 0.12$, $\sigma = 0.2$, $r = 0.05$; $\delta = 0.1$, $\gamma = 0.5$									
Unconstrained	39.2220	55.4683	67.9345	78.4440	87.7030	96.0739	103.7716	110.9365	117.6660	124.0308
Normal	39.2260	55.4739	67.9409	78.3797	85.1092	89.4669	92.9456	96.0641	99.0132	101.8670
T-distribution	39.2260	51.9595	57.5145	62.3511	66.9286	71.2870	75.4424	79.4122	83.2138	86.8639
Extreme Value	39.2260	55.4738	66.5307	72.0393	76.1240	79.8258	83.3675	86.8023	90.1442	93.3985

In summary, explicit solutions have been derived for the optimal portfolio behavior of investors with preferences assumed in Table 1 and for whom the CVaR constraint is binding. From the numerical results, the imposed of CVaR constraint don't impact the agent's propensity of consume severely. But, it will decrease the investment in risky assets. In addition, the non-normality asset loss distribution assumption on CVaR risk measure has more influence on asset allocation decision.

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