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A Novel Approach to Derivatives and Risk Less Modelling

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Abstract

In portfolio analysis, the major issue on the combined discussion with the derivatives and riskless option plays an important role in modeling. The stochastic differential equation derived on such model was given by Black and Scholes [2]. By a proper transformation of variables involved, this model can be put in as a harmonic equation. As the solution in this particular type has a barring in an abstract set up to equations and change of probability measures. A close connection between harmonic functions and martingales was initiated by Burkholder [3]. The background has suggested us a new direction of study in the derivatives and option pricing formula. We present here the basic concepts of derivatives, the connection between martingales and harmonic functions and the relevance of harmonic functions in the Black and Scholes model [6]. With these the real time problem can be viewed as special types of the abstract model developed here.

Keywords: Derivatives; martingales; Black and Scholes model; harmonic functions; probability measures

Basic Concepts

Derivatives

A simplest type of derivative is called a forward contract. In this one party agrees to buy a certain financial asset from another party for a fixed price at certain fixed data in future. At time K=0, the buyer of an option has to pay a premium to the writer of the option. The holder will exercise to write when asset at time $S_T > K$, a strike price. Then there are two basic issues that need to be addressed:

To determine a fair value of the premium that the buyer to pay the writer to price the option.

To device a trading strategy so that the writer of the option can generate an amount

Max $(S_T - K, 0)$ at time T, that is, to hedge the option. For that it is assumed that no profit without risk which is known as no arbitrage assumption.

No Arbitrage theorem

A market is arbitrage free if and only if there exists a probability measure $\hat{p}$ equivalent to p under which the discounted prices of assets are $\hat{p}$

Section 1

For a continuous model, let V_0 be the premium received by the writer of the option. Let us consider a portfolio consisting of one riskless asset B and a risky asset (stock) S. At time t=0, the value of the portfolio is V_0 where

$$V_0=\varphi_0^0 B_0+\varphi_0^1 S_0 \tag{1}$$

where φ_0^k $(k=0,1)$ denote the weights of the two financial assets in the portfolio at time t=0. Let us take $B_0=1$ and the risk free interest rate to be r (constant), then at time t

$$B_t=B_0\, e^{rt}=e^{rt} \tag{2}$$

Also, let S_t be the price of the stock at time t. Let us assume that the increment dS_t is governed by the stochastic equation

$$dS_t=\mu S_t dt+\sigma S_t d\beta_t \tag{3}$$

Then the random variable S_t is expressed as

$$S_t=S_0\exp\left[\,(\mu-\frac{\sigma^2}{2})\,t+\sigma\beta_t\right] \tag{4}$$

1.1. Eliminating the Drift Term [6]

Let us obtain a probability measure under which the discounted values of the stock $e^{rt}S_t$ and the option price $e^{-rt}C_t$ are martingales.

Consider $Z_t=e^{rt}S_t$ (5)

so that $dZ_t=(\mu-r)Z_t dt+\sigma Z_t d\beta_t$ (6)

For constant values of μ, r and σ , let us define a new Brownian motion $\hat{\beta}_t$ such that

$$\hat{\beta}_t=\left(\frac{\mu-r}{\sigma}\right)t+\beta_t \tag{7}$$

and $d\hat{\beta}_t=\left(\frac{\mu-r}{\sigma}\right)dt+d\beta_t$ (8)

Substituting for $d\beta_t$ in the equation of dZ_t , we get

$$dZ_t=\sigma Z_t d\hat{\beta}_t \tag{9}$$

1.2. Fair Arbitrage Free Price of the Option [6]

At time t ($0\le t\le T$), the value of the portfolio is

$$V_t=\varphi_{t\_}^0 e^{rt}+\varphi_{t\_}^1 S_t \tag{10}$$

where $V_t=\varphi_{t\_}^0$ and $\varphi_{t\_}^1$ are the weights and are determined by knowing the stock price S_t-dt at time $t-dt$ and they remain the same until the new price S_t is known. We can change the weights to φ_{t+}^k $(k=1,2)$. For a self- financing strategy, we need the condition

$$\varphi_{t\_}^0 e^{rt}+\varphi_{t\_}^1 S_t=\varphi_{t+}^0 e^{rt}+\varphi_{t+}^1 S_t \tag{11}$$

This implies that no wealth is added or removed while changing the weights.

The change in the value of the portfolio is

$$dV_t = dV_{t+dt} - V_t$$

$$= \varphi^0_{t+} e^{r(t+dt)} + \varphi^1_{t+} S_{t+dt} - \varphi^0_{t\_} e^{rt} + \varphi^1_{t\_} S \tag{12}$$

$$= \varphi^0_{t+} e^{r(t+dt)} + \varphi^1_{t+} S_{t+dt} - \varphi^0_{t+} e^{rt} + \varphi^1_{t+} S_t$$

$$= \varphi^0_{t+} e^{rt} (e^{rdt} - 1) + \varphi^1_{t+} (S_{t+dt} - S_t)$$

For infinitesimal dt, we get

$$dV_t = \varphi^0_{t+} r e^{rt} dt + \varphi^1_{t+} dS_t \tag{13}$$

Due to the self- financing strategy, the change dV_t is entirely coming from the change in the asset prices.

Defining the discounted value of the portfolio as $\hat{V}_t = e^{-rt} Vt$, then

$$d\hat{V}_t = -re^{-rt} dt Vt + e^{-rt} dVt$$

$$= -re^{-rt} dt(\ \varphi^1_{t+} S_t) + e^{-rt} (\varphi^1_{t+} dS_t) \tag{14}$$

$$= e^{-rt} \varphi^1_{t+} \left(-rS_t dt + \mu S_t dt + \sigma S_t d\beta_t\right)$$

$$d\hat{V}_t = e^{-rt} \varphi^1_{t+} \left((\mu - r) S_t dt + \sigma S_t d\beta_t\right)$$

We define a new Brownian motion process

$$\hat{\beta}_t = \left(\frac{\mu - r}{\sigma}\right) t + \beta_t$$

$$d\hat{\beta}_t = \left(\frac{\mu - r}{\sigma}\right) dt + d\beta_t$$

$$d\hat{V}_t = \varphi^1_{t+} \left(\sigma Z_t d\hat{\beta}_t\right) \tag{15}$$

where $Z_t = e^{-rt} S_t$

In terms of It$\hat{o}$ integral

$$\hat{V}_t = V_0 + \sigma \int_0^t \varphi^1_{t'} Z_{t'} d\hat{\beta}_{t'} \tag{16}$$

Since Z_t is a martingale, under the probability measure $\hat{p}$ defined by $\hat{\beta}_t$ the discounted value of the portfolio $\hat{V}_t$ is also a martingale under $\hat{p}$ .Consequently, we have

$$V_t = E^{\hat{p}} (e^{-r(T-t)} V_t \mid I_t) \tag{17}$$

We have assumed that there is a self-financing strategy which will replicate the claim $C_T = \max(S_T - K, 0)$ at time T by setting $C_T = V_T$. For $t \leq T$, we can write $C_t = V_t$. Since C_t is a function of S_t and t, then the discounted value $\hat{V}_t$ can be written as

$$\hat{V}_t = \hat{C}(Z_t, t) \tag{18}$$

Since $\hat{V}_t$ is a martingale under the measure $\hat{p}$, so is the option value $\hat{C}(Z_t, t)$

1.3. Black-Scholes Formula[6]

Using It$\hat{o}$'s lemma for $\hat{C}(Z_t, t)$, we get

$$\hat{C}(Z_t, t) = \hat{C}(Z_0, t) + \int_0^t \frac{\partial \hat{C}}{\partial Z_{t'}} dZ_t \tag{19}$$

This is because $\hat{C}$ is a martingale under the probability measure $\hat{p}$ and hence there cannot be any drift term.

From (9) we have

$$\hat{V}_t = V_0 + \int_0^t \varphi_{t'}^1 dZ_{t'} \quad = \hat{C}(Z_0, t) + \int_0^t \frac{\partial \hat{C}}{\partial Z_{t'}} dZ_{t'} \tag{20}$$

Comparing the terms in the above equation, we get

$$\varphi_t^1 = \frac{\partial \hat{C}}{\partial Z_t} \tag{21}$$

and $\varphi_t^0 = \hat{C}_t = \varphi_t^1 \hat{S}_t$ (22)

This is the delta hedging scheme.

Evaluating the integrals in the expression (17)

$$C_t = V_t = E^{\hat{p}}(e^{-r(T-t)} V_t \mid I_t) \tag{23}$$

With the final condition $C_T = \max(S_T - K, 0)$ using the probability measure $\hat{p}$.

Section 2

2.1. Harmonic Functions [3]

Let u be harmonic in the Euclidean half-space

$$R^{n+1} = \{ (x, y) : x \in R^n , y > 0 \}$$

and let $N_a = N_a(u)$ denote non tangential maximal function of u defined on R^n by

$N_a(u) = \sup \{ |u(s, y)| : (s, y) \in \Gamma_a(x) \}$

where $\Gamma_a(x) = \{ (s, y) : |x - s| < ay \text{ and } a > 0 \}$.

The area integral of n is the non-negative function $A_a = A_a(u)$ defined on R^n by

$$A_a^2(x) = \iint_{\Gamma_a(x)} |\nabla_u(s, y)|^2 \, y^{1-n} \, dyds$$

If h>0, let $N_{a,h}$ and $A_{a,h}$ be the truncated versions in which y is restricted to the interval 0<y<h. Let Q be the Cube in R^n of volume |Q| and m_q, the measure on the measurable subsets of R^n defined by

$m_q(E) = | E \cap Q |$

Let $\beta > 1, \delta > 1$, and suppose that the diameter of Q is 2ah. Then, for all $\lambda > 0$,

$$m_q(A_{a,h} > \beta\lambda, N_{2a,2h} \leq \delta\lambda) \leq \varepsilon_1 \; m_q(A_{a,h} > \lambda) \tag{24}$$

where $\varepsilon_1 = c\delta^2 / (\beta^2 - 1)$ and the choice of c depends only on n and a.

If u(q,h) = 0, where q is the centre of Q, then

$$m_q(A_{a,h} > \beta\lambda, A_{2a,2h} \leq \delta\lambda) \leq \varepsilon_2 \; m_q(N_{a,h} > \lambda) \tag{25}$$

2.2. Formulation of the Model

Let us select and fix a reference measure and associated expectation operator as p^* and V^*. Set of all predictable process such that the increasing process is integrable with respect to p^*. A trading strategy includes all predictable processes such that a trading strategy φ is said to be admissible if V^* is martingale under p^*. These martingales are closed under convex functions. Thus we are left to characterization of completeness with continuous trading involving fine structure of filtration. Thus every contingent claim is in some sense nearly attainable.

In this section, we set and prove a result connecting harmonic functions and martingales. We also give the connection between harmonic functions and market with no arbitrage pricing.

Theorem 2(a)

If u is harmonic in R^{n+1}, then

(i) $\int_{R^n} \varphi(A_a(u))\,dx \le c \int_{R^n} \varphi(N_a(u))\,dx$

and , provided $\lim_{Y\to\infty} u(0,Y)=0$,

(ii) $\int_{R^n} \varphi(N_a(u))\,dx \le c \int_{R^n} \varphi(A_a(u))\,dx$

In each case, the choice of c depends only on ϕ, n and a.

In particular, if $\lim_{Y\to\infty} u(0,Y)=0$, then

$$\| N_a(u) \|_p \approx \| A_a(u) \|_p \ ,0<p<\infty .$$

Proof

Using (24) and the theorem (1) of Burkholder (1979) and the fact that m_q is a finite measure, we obtain that

$$\int_{R^n} \varphi(A_{a,h})dm_Q \le c \int_{R^n} \varphi(N_{2a,2h})\,dm_Q$$

Therefore,

$$\int_{R^n} \varphi(A_{a,h})dx \le c \int_{R^n} \varphi(N_{2a,2h})\,dx$$

Where Q_h is chosen to increase with h and have diameter 2ah. By the monotone convergence theorem [7] ,

$$\int_{R^n} \varphi(A_a)dx \le c \int_{R^n} \varphi(N_{2a})\,dx$$

This completes the proof of (i).

The proof of (ii) is similar. Let Q_h be as above but with centre o. Then by (25)

$$\int_{Q_h} \varphi(N_{a,h}(u-u(0,h))dx \le c \int_{R^n} \varphi(A_{2a})\,dx$$

Using Fatou's lemma [7] and the fact that

$$\varphi(N_a) \le \liminf_{h\to\infty} \psi_h\, \varphi(N_{a,h}(u-u(0,h))$$

where ψ_h is the characteristic function of Q_h, we obtain

$$\int_{R^n} \varphi(N_a)dx \le c \int_{R^n} \varphi(A_{2a})\,dx$$

This gives (ii) with 2a in place of a and completes the proof of the theorem.

The following theorem describes the behavior of u near the boundary.

Theorem 2(b)

Suppose that u is harmonic in R^{n+1}. Then

$$\{N_{a,h}(u)<\infty\} = a.e \{A_{a,h}(u)<\infty\} .$$

Proof

$N_{2a,2h}$ is finite almost everywhere on the set where $N_{a,h}$ is finite. We have

$$m_q(A_{a,h}>\beta\lambda, N_{2a,2ah}\le\delta\lambda) \le \varepsilon_1 m_q(A_{a,h}>\lambda)$$

where $\varepsilon_1 = \dfrac{\varepsilon\delta^2}{\beta^2 - 1}$ and the choice of c depends only on n and a. Let $\beta \to \infty$ and that $\alpha \to \infty$ to obtain

$$m_q\,(A_{a,h} = \infty, N_{2a,2ah} < \infty) = 0\,.$$

This implies

$$\{N_{2a,2h} < \infty\} \underset{a.e}{\subset} \{A_{a,h} < \infty\}$$

This completes the proof of the theorem.

One of the important connections between martingales and harmonic functions is that the composition of the harmonic functions with Brownian motion is a martingale [4].

With the assumption of no arbitrage the change in the value of the risk free portfolio is

$$dV_t = rV_t\,dt \tag{26}$$

Substituting for dV_t and V_t we have

$$\left(\frac{\partial F}{\partial t}\right) + \frac{1}{2}\sigma^2 S_t^2 \left(\frac{\partial^2 F}{\partial t^2}\right) = r\left(F_t - \frac{\partial F}{\partial S} S_t\right)$$

and hence

$$\left(\frac{\partial F}{\partial t}\right) + rS_t\left(\frac{\partial F}{\partial S}\right) + \frac{1}{2}\sigma^2 S_t^2 \left(\frac{\partial^2 F}{\partial t^2}\right) - rF = 0 \tag{27}$$

which is the Black-Scholes pde.

If K is the strike price and T the expiration time, then at time T, the solution of the above pde must satisfy the condition

$$F\,(S_T\,, T) = \max\,(S_T - K\,,\,0) \tag{28}$$

The pde was solved by Black and Scholes with this condition to obtain the option price at time t. This equation can be transformed into a harmonic function. Thus connecting harmonic function and martingale providing us a solution to option pricing formula.

Conclusion

We presented an expository article to model derivatives and its relevance to option pricing model through probability measures and martingales.

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