Construction of Pde Black - Scholes with Jump-Diffusion Models

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Abstract

Jump diffusion is a stochastic process that involves jumps and diffusion. It has important applications in magnetic reconnection, coronal mass ejections, condensed matter physics, in Pattern theory and computational vision and in option pricing. In option pricing, a jump-diffusion model is a form of mixture model, mixing a jump process and a diffusion process. Robert C. Merton as an extension of jump models has introduced jump-diffusion models. Due to their computational tractability, the special case of a basic affine jump diffusion is popular for some credit risk and short-rate models. Jump diffusion processes have been used in modern finance to capture discontinuous behavior in asset pricing. The jump-diffusion process was constructed to have ergodic properties so that after initially flowing away from its initial condition it would generate samples from the posterior probability model. The main objective of this study is to Construction of Black - Scholes PDE with Jump-Diffusion Models. The main goal of this study is fourfold: 1) First, we begin our approach to brief contemporary for the Stochastic models for option pricing- Jump-diffusion model. 2) Next we extent this approach to introduce the fundamental derivations of the Black Scholes model for the pricing of a European call option. 3) Then we reinterpret the Black Scholes in the formalism of jump process also consider the problem of pricing a European call when the underlying asset is jump process. 4) Finally, we construct the mathematical model for Black – Scholes PDE equation with Jump diffusion process. In addition, this paper ends with conclusion.

Keywords: Stochastic Process, Brownian motion, Option Pricing, Black – Scholes Model, Partial Differential Equation and Jump-Diffusion Models.

JEL Classification:

1. Introduction

The Jump Diffusion Process, which describes the movements in the price of an underlying as simply neither, jumps nor defined as a pure diffusion process. The price movements are, rather, represented as a jump followed by continuous diffusion. The combined dynamics have been incorporated into option pricing models, though still are there some implementation problems and practical difficulties hindering the emergence of a full-fledge jump diffusion model.

A valuation model that allows for jumps in underlying assets’ prices superimposed on to a diffusion process such as geometric Brownian motion. In Black-Scholes option pricing model, trading is assumed to take place continuously in time, while the underlying asset price follows a continuous sample path. However, occasional jumps in asset price are commonplace in asset price dynamics. Such jumps may reflect the release of new important information on a given firm or industry or even local
economy. Modeling of the asset price process by combining normal price fluctuation and abnormal jumps was introduced by [21]. The geometric Brownian process models the normal fluctuation, while the jumps are modeled by Poisson distributed events, where jump events are assumed independent and identically distributed models.

The ‘diffusion’ part of the nomenclature refers to the fact that these processes can have a Brownian motion component or, more generally, an integral with respect to Brownian motion. In addition, the paths of these processes may have jumps. One can also construct processes in which there are infinitely many jumps in a finite time interval, although for such processes it is necessarily the case that, for each positive threshold, only finitely many jumps can have a size exceeding the threshold in any finite time interval.

2. Stochastic Models for Option Pricing- Jump-Diffusion Model

The derivative pricing model developed by Black, Scholes and Merton is a huge success in financial engineering area. It says that there exists an arbitrage-free price for plain vanilla options and the investors can perfectly hedge them by constructing a self-finance portfolio. However, the empirical observation demonstrates that this model is not perfect. For one thing, two different options on the same underlying with the same expiry date but different strike prices can imply different volatility. Indeed, if one plots the implied volatility as a function of the strike price of an option, the curve is roughly smile-shaped. For another thing, the stock and foreign exchange prices are simply not log-normally distributed as the model assumes. In addition, in fact, the actual distribution of the logs of asset price changes have fat tails. To cope with these problems, we need to introduce models that are more sophisticated.

A big shortcoming of Black-Scholes model is that it assumes the asset price is a continuous function. Nevertheless, in reality, the stock market undergoes crash periodically. We, therefore, wish to permit the possibility of jumps in our model. In this post, we briefly discuss the jump-diffusion model presented by Merton. In addition, in order to illustrate it, we first briefly discuss the properties of Poisson process.

The two basic building blocks of every jump – diffusion model are the Brownian motion (the diffusion part) and the Poisson process (the jump part). Brownian motion is a familiar object to every option trader since the appearance of the Black-Scholes model, but a few words about the Poisson process are in order.

Take a sequence \( \{ \tau_i \} \) of independent exponential random variables with parameter \( \lambda \), that is, with cumulative distribution function \( P[\tau_i \geq y] = e^{-\lambda y} \) and let \( T_n = \sum_{i=1}^{n} \tau_i \). The process

\[
N_t = \sum_{n \geq 1} 1_{\tau_n \leq t}
\]

(1)

Is called the Poisson process with parameter \( \lambda \). The jumps occur at times \( T_i \) and the intervals between jumps (the waiting times) are exponentially distributed. At every date \( t > 0 \), \( N_t \) has the Poisson distribution with parameter \( \lambda t \), that is, it is integer-valued and

\[
P(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad n = 0,1,...
\]

(2)

The Poisson process shares with the Brownian motion the very important property of independence and stationarity of increments, that is, for every \( t > s \) the increment \( N_t - N_s \) is independent from the history of the process up to time \( s \) and has the same law as \( N_{t-s} \).

The expectation of the increment is
\[ E[N(t) - N(s)] = \lambda(t - s) \]  

(3)

In addition, the variance can derive from the same

\[ \text{Var}[N(t) - N(s)] = \lambda(t - s) \]  

(4)

Now we can define the compensated Poisson process as

\[ M(t) = N(t) - \lambda t \]  

(5)

Then \( M(t) \) is the martingale.

The processes with independent and stationary increments are called Levy processes after the French mathematician Paul Levy. The notion of characteristic function of a random variable plays an essential role in the study of jump-diffusion processes: often we do not know the distribution function of such a process in closed form but the characteristic function is known explicitly. The characteristic function of a random variable \( X \) is defined by

\[ \phi_X(u) = E[e^{iuX}] \]  

(6)

For the Poisson process, this gives

\[ E[e^{iuN_t}] = e^{t \{e^{iu} - 1\}} \]  

(7)

Here, the computation can be done directly using equation (2). The stock price will be modelled by the stochastic differential equation and for financial applications, it is of little interest to have a process with a single possible jump size. The compound Poisson process is a generalization where the waiting times between jumps are exponential but the jump sizes can have an arbitrary distribution. More precisely, let \( N \) be a Poisson process with parameter \( \lambda \) and \( \{Y_i\}_{i=1} \) be a sequence of independent random variables with law \( f \). The process

\[ X_t = \sum_{i=1}^{N_t} Y_i \]  

(8)

is called compound Poisson process. The compound Poisson process has independent and stationary increments. Its law at a given time \( t \) is not known explicitly but the characteristic function is known and has the form

\[ E[e^{iuX_t}] = e^{t \{\int_0^t e^{iu}f(ds)\}} \]  

(9)

Simulation of compound Poisson process Contrary to more complex jump processes, the compound Poisson process is easy to simulate. The algorithm is based on the following fact that conditionally on \( N_T = n \), the jump times \( T_1, \ldots, T_n \) of a Poisson process on the interval \([0, T]\) are distributed as \( n \) independent ordered uniform on \([0, T]\) and this fact that leads to the following algorithm.

1. Simulate \( N_T \) from the Poisson distribution with parameter \( \lambda t \).
2. Simulate \( N_T \) uniform random variables \( \{U_i\}_{i=1}^{N_T} \) on \([0, T]\).
3. Simulate \( N_T \) independent variables \( \{Y_i\}_{i=1}^{N_T} \) with law \( f \).
4. The process is given by

\[ X_t = \sum_{i=1}^{N_T} Y_i 1_{U_i \leq t} \]
Jump-diffusions and Levy processes combining a Brownian motion with drift and a compound Poisson process, we obtain the simplest case of a jump diffusion - a process that sometimes jumps and has a continuous but random evolution between the jump times

\[ X_t = \mu t + \sigma B_t + \sum_{i=1}^{N_t} Y_i \]  \hspace{1cm} (10)

The best-known model of this type in finance is the Merton model \([21]\), where the stock price is \( S_t = S_0 e^{X_t} \) with \( X_t \) as above and the jumps \( \{ Y_i \} \) have Gaussian distribution.

The process (10) is again a Levy process and its characteristic function can be computed by multiplying the CF of the Brownian motion and that of the compound Poisson process (since the two parts are independent):

\[
E[e^{iuX_t}] = e^{\left\{ i\mu u + \frac{\sigma^2 u^2}{2} + \lambda \left[ e^{iuY} - 1 \right]/(iu) \right\}} \hspace{1cm} (11)
\]

The class of Levy processes is not limited to jump-diffusions of the form (2): there exist Levy processes with infinitely many jumps in every interval. Most of such jumps are very small and there is only a finite number of jumps with absolute value greater than any given positive number. One of the simplest examples of this kind is the gamma process, a process with independent and stationary increments such that for all \( t \), the law \( p_t \) of \( X_t \) is the gamma law with parameters \( \lambda \) and \( ct \):

\[
p_t = \frac{\lambda^c t}{\Gamma(ct)} x^{ct-1} e^{-\lambda x} \hspace{1cm} (12)
\]

The gamma process is an increasing Levy process (also called subordinator). Its characteristic function has a very simple form:

\[
E[e^{iuX_t}] = \left( 1 - \frac{iu}{\lambda} \right)^{-ct} \hspace{1cm} (13)
\]

The gamma process is the building block for a very popular jump model, the variance gamma process \([20, 19]\), which is constructed by taking a Brownian motion with drift and changing its time scale with a gamma process:

\[ Y_t = \mu X_t + \sigma B_t \] \hspace{1cm} (14)

Using \( Y_t \) to model the logarithm of stock prices can be justified by saying that the price is a geometric Brownian motion if viewed on a stochastic time scale given by the gamma process \([16]\). The variance gamma process is another example of a Levy process with infinitely many jumps and has characteristic function

\[
E(e^{iuY_t}) = \left( 1 + \frac{\sigma^2 u^2}{2} - i\mu ku \right)^{-kt} \hspace{1cm} (15)
\]

The parameters have the following (approximate) interpretation: \( \sigma \) is the variance parameter, \( \mu \) is the skewness parameter and \( k \) is responsible for the kurtosis of the process.

In general, every Levy process can be represented in the form

\[ X_t = \lambda t + \sigma B_t + Z_t \] \hspace{1cm} (16)

Where \( Z_t \) is a jump process with (possibly) infinitely many jumps. A detailed description of this component is given by the Levy - Ito decomposition that is beyond the scope of this introductory paper. The characteristic function of a Levy process is given by the Lévy-Khintchine formula:
$$E(e^{iaX_t}) = e^{\left(-\frac{1}{2}a^2\nu(dx) + \frac{i}{2}a\mu(dx) + \frac{i}{2}a^2\nu(dx)\right)}$$  \hfill (17)$$

Where \( \nu \) is a positive measure in \( R \), description the jump of the process: the Levy measure. If \( X \) is compounded Poisson, then \( \nu(R) < \infty \) and \( \nu(dx) = \lambda f(dx) \) but in the general case \( \nu \) need not be a finite measure.

It must satisfy the constraint

$$\int_R \left(1 + x^2\right) \nu(dx) < \infty$$  \hfill (18)$$

and describes the jumps of \( X \) in the following sense: for every closed set \( A \subset R \) with \( 0 \notin A \), \( \nu(A) \) is the average number of jumps of \( X \) in the time interval \([0,T]\), whose sizes fall in \( A \).

To keep the discussion simple, in the rest of this section we will only consider Levy jump-diffusions, that is, Levy processes with finite jump intensity of the form (10), but with the new notation

$$\nu(dx) = f(dx)$$

for the Levy measure.

The characteristic function of such a process therefore takes the form

$$E(e^{iaX_t}) = e^{\left(-\frac{1}{2}a^2\nu(dx) + \frac{i}{2}a\mu(dx) + \frac{i}{2}a^2\nu(dx)\right)}$$  \hfill (19)$$

Exponential Levy models to ensure positivity as well as the independence and stationarity of log-returns, stock prices are usually modeled as exponentials of Levy processes:

$$S_t = S_0 e^{\tilde{X}_t}$$  \hfill (20)$$

In the jump – diffusion case this gives

$$S_t = S_0 e^{X_t} = S_0 \left[\mu t + \sigma dB_t + \sum_{i=1}^{N(t)} Y_i\right]$$  \hfill (21)$$

Between the jumps, the process evolves like a geometric Brownian motion, and after each jump, the value of \( S_t \) is multiplied by \( e^{Y_i} \). This model can see as a generalization of the Black-Scholes model:

$$dS_t = \tilde{\mu} dt + \sigma dB_t + dJ_t$$  \hfill (22)$$

Here, \( J_t \) is a compound Poisson process such that the \( i \)-th jump of \( J \) is equal to \( e^{Y_i} \). For instance, if \( Y_t \) have Gaussian distribution, \( S \) will have lognormally distributed jumps.

The notation \( S_t \) – means that whenever there is a jump, the value of the process before the jump is used on the left-hand side of the formula. The forms (20) and (22) are equivalent: for a model of the first kind, one can always find a model of the second kind with the same law. In the rest of the paper, unless explicitly stated otherwise, we will use the exponential form (20).

For option pricing, we will explicitly include the interest rate into the definition of the exponential Levy model:

$$S_t = S_0 e^{\tilde{X}_t}$$  \hfill (23)$$

While the forms (20) and (23) are equivalent, the second one leads to a slightly simpler notation. In this case, under the risk-neutral probability, \( e^{X_t} \) must be a martingale and from the Levy - Khintchine formula (19) combined with the independent increments property we conclude that this is the case if
\( b + \frac{\sigma^2}{2} + \int_{R} (e^x - 1) \nu(dx) = 0 \) \hspace{1cm} (24)

The model (23) admits no arbitrage opportunity if there exists an equivalent probability under which \( e^{X_t} \) is a martingale. For Levy processes, it can be shown that this is usually the case, namely an exponential Levy model is arbitrage-free if and only if the trajectories of \( X \) are not almost surely increasing nor almost surely decreasing.

If a Brownian component is present, the martingale probability can be obtained by changing the drift as in the Black-Scholes setting. Otherwise, in finite-intensity models, the drift must remain fixed under all equivalent probabilities since it can be observed from a single stock price trajectory. To satisfy the martingale constraint (24), one must therefore change the Levy measure, i.e. the intensity of jumps. To understand how this works, suppose that \( X \) is a Poisson process with drift:

\[ X_t = N_t - at, \quad a > 0. \] \hspace{1cm} (25)

We can obtain a martingale probability by changing the intensity of \( N \) to \( \lambda_{\text{mart}} = \frac{a}{e-1} \). If, however, \( X \) is a Poisson process without drift (increasing trajectories), one cannot find a value of \( \lambda > 0 \) for which \( e^{X_t} \) is a martingale.

Although the class of Levy processes is quite rich, it is sometimes insufficient for multi period financial modeling for the following reasons:

- Due to the stationarity of increments, the stock price returns for a fixed time horizon always have the same law. It is therefore impossible to incorporate any kind of new market information into the return distribution.
- For a Levy process, the law of \( X_t \) for any given time horizon \( t \) is completely determined by the law of \( X_1 \). Therefore, moments and cumulates depend on time in a well-defined manner which does not always coincide with the empirically observed time dependence of these quantities [3].

For these reasons, several models combining jumps and stochastic volatility appeared in the literature. In the Bates [2] model, one of the most popular examples of the class, an independent jump component is added to the Heston stochastic volatility model:

\[ dX_t = \mu dt + \sqrt{V_t} dW^X_t + dZ_t, \quad S_t = S_0 e^{X_t} \] \hspace{1cm} (26)

\[ dV_t = \xi (\eta - V_t) dt + \theta \sqrt{V_t} dW^V_t \quad d\langle W^V, W^X \rangle_t = \rho dt \] \hspace{1cm} (27)

where \( Z \) is a compound Poisson process with Gaussian jumps.

### 2. The Black Scholes Model

In order to facilitate continuity and elucidate various concepts that will be used in the sequel, we give below two fundamental derivations of the Black Scholes model for the pricing of an European call option. The European call option is a financial contingent claim that entails a right (but not an obligation) to the holder of the option to buy one unit of the underlying asset at a future date (called the exercise date or maturity date) at a price (called the exercise price). The option contract, therefore, has a terminal payoff of \( \max(S_T - E, 0) = (S_T - E)^+ \) where \( S_T \) is the stock price on the exercise date and \( E \) is the exercise price.

We consider a non dividend paying stock, the price process of which follows the geometric Brownian motion with drift \( S_t = e^{(\mu - \sigma \omega^p)} \). The logarithm of the stock price \( Y_t = \ln S_t \) follows the stochastic differential equation

\[ dY_t = \mu dt + \sigma dW^p_t \] \hspace{1cm} (28)
where $\mu$ and $\sigma$ are constants representing the long term drift and the noisiness (diffusion) respectively in the stock price and $W_t^p$ is a regular Brownian motion representing Gaussian white noise with zero mean and $\delta$ correlation in time i.e. $E^p(dW_t,dW_{t'}) = dt dt' \delta(t - t')$ and on some filtered probability space $(\Omega, (F_t), P)$. Application of Ito’s formula yields the following SDE for the stock price process

$$dS_t = \left(\mu + \frac{1}{2} \sigma^2\right)S_t dt + \sigma S_t dW_t^p$$

(29)

Let $C(S_t, t)$ denote the instantaneous price of a call option with exercise price $E$ at any time $t$ before maturity when the price per unit of the underlying is $S_t$. We assume that $C(S_t, t)$ does not depend on the past price history of the underlying. Applying the Ito formula to $C(S_t, t)$ yields

$$dC_t = (\mu S_t \frac{\partial C_t}{\partial S_t} + \frac{1}{2} \sigma^2 S_t \frac{\partial^2 C_t}{\partial S_t^2} + \frac{1}{2} \frac{\partial C_t}{\partial t} S_t + \sigma S_t \frac{\partial C_t}{\partial S_t} dW_t^p) dt + \frac{\partial C_t}{\partial S_t} dS_t dW_t^p, \quad (30)$$

The first approach to the Black Scholes formula attempts to remove the randomness in the previously mentioned formula by constituting a hedge in the form of another random process correlated to the above price process. For this purpose, we introduce a ‘bond’ in our model that evolves according to the following price process

$$dB_t = r dt, B_t = 1,$$

(31)

where $r$ is the relevant risk free interest rate.

Making use of $\phi_t$ units of the underlying asset and $\psi_t$ units of the bond, where $\phi_t = \frac{\partial C(S_t, t)}{\partial S_t}$, $B_t \psi_t = C(S_t, t) - \phi_t S_t$, we can now construct a trading strategy that has the following properties

It exactly replicates the price process of the call option i.e.

$$\phi_t S_t + \psi_t B_t = C(S_t, t), \forall t \in [0, T], \quad (32)$$

It is self-financing i.e. $\phi_t dS_t + \psi_t dB_t = dV_t, \forall t \in [0, T].$  

(33)

Using eqs. (28), (30), (32) & (33) we have

$$dC = \left(\phi_t \mu S_t + \frac{1}{2} \phi_t \sigma^2 S_t + \psi_t r B_t\right) dt + \phi_t \sigma S_t dW_t^p.$$  

(34)

Matching the diffusion terms of (30) & (34) and using (32), we get the previously mentioned expressions for $\phi_t$ and $\psi_t$, respectively. Further, a comparison of the drift terms yields

$$r S_t \frac{\partial C(S_t, t)}{\partial S_t} + \frac{1}{2} \sigma^2 S_t \frac{\partial^2 C(S_t, t)}{\partial S_t^2} + \frac{\partial C(S_t, t)}{\partial t} = r C(S_t, t).$$

(35)

This is the fundamental PDE for asset pricing. For the pricing of the call option, it must additionally satisfy the boundary condition $C(S_T, T) = \max(S_T - E, 0)$, which leads to the following solution

$$C(x, t) = x \Phi(z) - e^{-r(T-t)} \Phi\left(z - \sigma \sqrt{T-t}\right), \quad x = S_t$$

(36)
\[
\begin{align*}
    z &= \frac{\ln \left( \frac{S}{K} \right) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} \\
    \phi(z) &= \Phi(z)
\end{align*}
\]

and \( \Phi(z) \) denotes the cumulative density of the standard normal distribution. Eqs. (36), (38) represent the celebrated Black Scholes formula for the instantaneous price of a call option.

The second approach to the Black Scholes formula [3 1] is more rigorous and does not depend on the Markov property of the stock price. Applying Girsanov’s theorem to the price process (29), we perform a change of measure and define a probability measure \( Q \) such that the discounted stock price process \( Z_t = S_t e^{-rt} \) or equivalently

\[
dZ_t = \left( \mu - r + \frac{1}{2} \sigma^2 \right) Z_t dt + \sigma Z_t dW^p_t
\]

behaves as a martingale with respect to \( Q \). This is performed by eliminating the drift term through the transformation

\[
\frac{\mu - r + \frac{1}{2} \sigma^2}{\sigma} \to \gamma = \gamma.
\]

Whence \( W_t^Q = W_t^p + \gamma \) is a Brownian motion without drift with respect to the measure \( Q \) and \( \sigma Z_t dW^Q_t \) which is drift less under the measure \( Q \) and hence, \( Z_t \) is a \( Q \) martingale. The two measures \( P \) & \( Q \) are related through the Radon Nikodym derivative i.e.

\[
\xi(t) = \frac{dQ}{dP} = \exp \left( -\gamma W^p_t - \frac{1}{2} \gamma^2 t \right)
\]

and the expectation operators under the two measures are related as

\[
E^Q \left[ X_t | F_t \right] = \xi^{-1}(s) E^P \left[ \xi(t) X_s | F_t \right]
\]

Our next step in martingale-based pricing is to constitute a \( Q \) martingale process that hits the discounted value of the contingent claim i.e. call option. This is formed by taking the conditional expectation of the discounted terminal payoff from the claim under the \( Q \) measure i.e.

\[
E^Q \left[ e^{-r(T-t)} (S_T - E)^+ | F_t \right].
\]

As in the PDE approach, we now constitute a self-financing portfolio of \( \phi_t \) units of the underlying asset and \( \psi_t \) units of the bond, where

\[
\phi_t = \frac{\partial C}{\partial S} (S_t, t), \quad B_t = C(S_t, t) - \phi_t S_t,
\]

which replicates the payoff of the claim at all points in time. The value of this portfolio at any time \( t \) can be shown to be equal to \( V_t = e^{r^p} E_t \) with \( E_t \) being given by eq.(51). It follows that the value of the replicating portfolio and hence of the call option at time \( t \) is given by

\[
V_t = e^{r^p} E_t = e^{-r(T-t)} E^Q \left[ (S_T - E)^+ | F_t \right]
\]

The expectation value of the contingent claim \( \max(S_T - E, 0) = (S_T - E)^+ \) under the measure \( Q \) depends only on the marginal distribution of the stock price process \( S_t \) under the measure \( Q \), which is obtained by writing it in terms of \( Q \) Brownian motion \( W^Q_t \). We have, from eq.(38),
\[ d(In S_t) = \mu dt + \sigma dW_t^p = \left( r - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t^Q \]

which on integration yields

\[ S_t = S_0 \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma W_t^Q \right]. \quad (45) \]

This is a \( S_0 \) scaled normal with mean and variance \( \left( r - \frac{1}{2} \sigma^2 \right) t, \sigma^2 t \) respectively. Writing

\[ Z = N\left( -\frac{1}{2} \sigma^2 T, \sigma^2 T \right) \], we have \( V_t = e^{-r(T-t)} E^Q \left[ S_0 e^{Z_{T-t}} - E \right] \].

(46)

This equals

\[ \int_{h_{(E/F_t)}} \left( S_0 e^{x} - E e^{-r(T-t)} \right) \exp \left( -\left( x + \frac{1}{2} \sigma^2 T \right)^2 \right) dx \quad (47) \]

This integral can be decomposed by a change of variables to a pair of standard cumulative normal integrals, which yield the Black Scholes formula (36).

### 3. Pricing A European Call In A Jump Model

In this section, we consider the problem of pricing a European call when the underlying asset is a jump process. Here we are work out the underlying asset is driven by a single Poisson process. Now, we can rewrite the Poisson process with following notation.

Ley \( N(t) \) be a Poisson process on a probability space \( (\Omega, F, P) \) relative to a filtration \( F(t), t \geq 0 \). We denote the intensity of \( N(t) \) by \( \lambda \), a positive constant.

The compensated Poisson process \( M(t) = N(t) - \lambda t \) is a martingale under \( P \). Let \( \tilde{\lambda} \) be a positive number. We define

\[ Z(t) = e^{(\tilde{\lambda} - \lambda)t} \left( \frac{\tilde{\lambda}}{\lambda} \right)^{N(t)} \]

(48)

We fix a time \( T > 0 \) and will use \( Z(t) \) to change to a new measure \( \tilde{P} \) under which \( N(t), 0 \leq t \leq T \), has intensity \( \tilde{\lambda} \) rather than \( \lambda \).

Differentiate the equation [48] and we get

\[ dZ(t) = \frac{\tilde{\lambda} - \lambda}{\lambda} Z(t-) dM(t) \]

(49)

In particular, \( Z(t) \) is a martingale under \( P \) and \( EZ(t) = 1 \) for all \( t \).

Now, consider stock modeled as a geometric Poisson process

\[ S(t) = S(0) e^{(\alpha + N(t)) \log(\sigma + 1) - \lambda t - \frac{1}{2} \sigma^2 t} \]

\[ S(0) = e^{(\alpha - \lambda \sigma)t} (\sigma + 1)^{N(t)} \]

(50)
Where \( \sigma > -1, \sigma \neq 0 \), and \( N(t) \) is a Poisson process with intensity \( \lambda \) under the actual probability measure \( \mathbb{P} \). where \( e^{-\sigma S(t)} \) is a martingale under \( \mathbb{P} \) hence \( S(t) \) has mean rate of return \( \alpha \).

Now the return in which the underlying asset price is given in equation (50) and which differential is
\[
dS(t) = \alpha S(t)dt + \sigma S(t)\ dM(t) \tag{51}
\]
Now we fix a positive time \( T \) and wish to price a European call whose payoff at time \( T \) is
\[
V(T) = (S(T) - K)^+
\tag{52}
\]
We saw from the equation (50) and assume \( \lambda > \frac{\alpha - r}{\sigma} \) in order to rule out arbitrage. Then
\[
\tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}
\tag{53}
\]
Is positive, and there is a risk – neutral measure given by \( \tilde{\mathbb{P}}(A) = \int_A Z(t) d\mathbb{P} \) for all \( A \in \mathcal{F} \).

\[
Z(t) = e^{\{\tilde{\lambda} - \frac{1}{2} \tilde{\lambda}^2\} \ N(t)}
\]
Where
Under the risk – neutral measure, the compensated Poisson process \( \tilde{M}(t) = N(t) - \tilde{\lambda} t \) is a martingale, and
\[
dS(t) = rS(t)dt + \sigma S(t)\ d\tilde{M}(t) \tag{54}
\]
Or, equivalently,
\[
d(e^{-\sigma S(t)}) = \sigma e^{-\sigma S(t)} \ d\tilde{M}(t) \tag{55}
\]
The discounted asset price is a martingale under \( \tilde{\mathbb{P}} \). In terms of \( \tilde{\lambda} \), we may rewrite the (55)
\[
S(t) = S(0)e^{\{r-\tilde{\lambda}\sigma\}(\sigma+1)^{N(t)}} \tag{56}
\]
For \( 0 \leq t \leq T \), let \( V(t) \) denote the risk – neutral price of a European call paying \( V(T) = (S(T) - K)^+ \) at time \( T \). The discount call price is a martingale under the risk – neutral measure. And the call price \( V(t) \) satisfies
\[
e^{-rt}V(t) = \tilde{E}[e^{-rT}V(T) \mid F(t)] = \tilde{E}[e^{-rT}(S(T) - K)^+ \mid F(t)] \tag{57}
\]
We have
\[
S(T) = S(0)e^{\{r-\tilde{\lambda}\sigma\}(\sigma+1)^{N(t)}} . e^{\{r-\tilde{\lambda}\sigma\}(T-t)}(\sigma+1)^{N(T)-N(t)}
\]
\[
= S(t) . e^{\{r-\tilde{\lambda}\sigma\}(T-t)}(\sigma+1)^{N(T)-N(t)} \tag{58}
\]
And it follows that
\[
V(t) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^+ \mid F(t)]
\]
The random variable $S(t)$ is $F(t)$ - measurable, whereas $e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)}$ is independent $F(t)$. According to the independent Lemma. 

\[ V(t) = c(t, S(t)), \] (60)

Where

\[ c(t, x) = \tilde{E} \left[ e^{-r(T-t)} \left( x e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)} - K \right)^+ \right] \] (59)

\[ = \sum_{j=0}^{\infty} e^{-r(T-t)} \left( x e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^j - K \right)^+ \tilde{\lambda}^j (T-t)^{j+1} j! e^{-\tilde{\lambda}(T-t)} \] (61)

Then simplify this equation (61)

\[ c(t, x) = \sum_{j=0}^{\infty} \left( x e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^j - K e^{-r(T-t)} \right)^+ \tilde{\lambda}^j (T-t)^{j+1} j! e^{-\tilde{\lambda}(T-t)} \] (62)

The risk – neutral price of the call $c(t, x)$ can be computed. Then $j = 0$. Then. We can rewrite the equation (62)

\[ c(t, x) = \left( x e^{(r-\tilde{\lambda}\sigma)(T-t)} - K e^{-r(T-t)} \right)^+ e^{-\tilde{\lambda}(T-t)} \] (63)

When $t = T$, this term is $(x - K)^+$, and it is only nonzero term in the cum in equation (62) when $t = T$. Therefore, the function $c$ satisfies the terminal condition

\[ c(T, x) = (x - K)^+ \text{ for all } x \geq 0. \] (64)

Next we derive the PDE that $c(t, x)$ must satisfy.

The useful iterated conditioning argument shows that

\[ e^{-rt} c(t, S(t)) = e^{-rt} V(t) = \tilde{E} \left[ e^{-rT} (S(T) - K)^+ \right] \] (64)

Is a martingale under $\tilde{P}$. Therefore, we compute $d(e^{-rt} c(t, S(t)))$ and set the $dt$ term equal to zero. The stochastic differential equation (54) may be rewrite as

\[ dS(t) = (r - \tilde{\lambda}\sigma)S(t) dt + \sigma S(t) dN(t) \] (65)

Which shows that the continuous part of the stock price satisfies

\[ dS^c(t) = (r - \tilde{\lambda}\sigma)S(t) dt \] (66)

On the other pointer, if the stock price jump at time $t$, then

\[ \Delta S(t) = S(t) - S(t-) = \sigma S(t-) \quad S(t) = (\sigma+1)S(t-) \] (67)

The Ito – Doeblin formula implies

\[ e^{-rt} c(t, S(t)) = c(0, S(0)) + \int_0^t e^{-ru} \left[ -rc(u, S(u))du + c(u, S(u))du \right] \]

\[ + \sum_{0 < u < t} e^{-ru} [c(u, S(u)) - c(u, S(u-))] \] (68)
\[ c(0, S(0)) + \int_0^t e^{-ru} \left[ -rc(u, S(u)) + c_t(u, S(u)) + (r - \tilde{\alpha})S(u)c_x(u, S(u)) \right] du \\
+ \int_0^t e^{-ru} \left[ c_t(u, \sigma + 1)S(u) - c_t(u, S(u)) \right] dN(u) \]

\[ = c(0, S(0)) + \int_0^t e^{-ru} \left[ -rc(u, S(u)) + c_t(u, S(u)) + (r - \tilde{\alpha})S(u)c_x(u, S(u)) \right] du \\
+ \int_0^t e^{-ru} \left[ c_t(u, \sigma + 1)S(u) - c_t(u, S(u)) \right] d\tilde{M}(u) \]

However, the integral
\[ \int_0^t e^{-ru} \left[ c_t(u, \sigma + 1)S(u) - c_t(u, S(u)) \right] d\tilde{M}(u). \]

Is the same as the integral
\[ \int_0^t e^{-ru} \left[ c_t(u, \sigma + 1)S(u) - c_t(u, S(u)) \right] d\tilde{M}(u). \]

We have shown that
\[ e^{-rt}c(t, S(t)) = c(0, S(0)) + \int_0^t e^{-ru} \left[ -rc(u, S(u)) + c_t(u, S(u)) + (r - \tilde{\alpha})S(u)c_x(u, S(u)) \right] du \\
+ \int_0^t e^{-ru} \left[ c_t(u, \sigma + 1)S(u) - c_t(u, S(u)) \right] d\tilde{M}(u) \]

The last integral is a martingale because the integrator \( \tilde{M}(u) \) is a martingale and the integrated is left – continuous. Because left side of equation (73), is also a martingale we can then solve for
\[ c(0, S(0)) + \int_0^t e^{-ru} \left[ -rc(u, S(u)) + c_t(u, S(u)) + (r - \tilde{\alpha})S(u)c_x(u, S(u)) \right] du \\
+ \tilde{\lambda} \left( c_t(u, \sigma + 1)S(u) - c_t(u, S(u)) \right) du \]

And it is difference of two martingales and hence is itself martingales. This can only happen if the integrand is zero.
\[ -rc(t, S(t)) + c_t(t, S(t)) + (r - \tilde{\alpha})S(t)c_x(t, S(t)) \\
+ \tilde{\lambda} \left( c_t(t, \sigma + 1)S(t) - c_t(t, S(t)) \right) = 0 \]

From the equation (75) and (73) and take the first taking the differential in equation (73)
\[ d\left( e^{-rt}c(t, S(t)) \right) = e^{-rt} \left[ -rc(t, S(t)) + c_t(t, S(t)) + (r - \tilde{\alpha})S(t)c_x(t, S(t)) \right] dt \\
+ \tilde{\lambda} \left( c_t(t, \sigma + 1)S(t) - c_t(t, S(t)) \right) d\tilde{M}(t) \]

and then setting the \( dt \) term equal to zero and the non \( dt \) term has a martingale integrator, and if this integrator has jumps, then the integrand fore this martingale is left – continuous. In particular, we have
\[
\begin{align*}
d(e^{-rt}c(t,S(t))) &= e^{-rt} \left[ -rc(t,S(t)) + c(t,S(t)) + (r - \tilde{\lambda}\sigma)c_t(t,S(t)) \right] dt \\
&\quad + e^{-rt} \left[ c(t,(\sigma + 1)S(t)) - c(t,S(u - )) \right]dN(t) \\
\end{align*}
\]

(77)

But setting the \(dt\) term
\[
\begin{align*}
e^{-rt} \left[ -rc(t,S(t)) + c(t,S(t)) + (r - \tilde{\lambda}\sigma)c_t(t,S(t)) \right] dt \\
\end{align*}
\]

(78)

In this expression equal to zero gives an incorrect result because the non-\(dt\) term has integrator \(dN(t)\) and \(N(t)\) is not a martingale.

By replacing the stock price process \(S(t)\) in equation (75) by a dummy variable \(x\). This gives the equation
\[
\begin{align*}
-rc(t,x) + c_x(t,x) + (r - \tilde{\lambda}\sigma)c_t(t,x) + \tilde{\lambda}(c(t,(\sigma + 1)x) - c(t,x)) = 0
\end{align*}
\]

(79)

Which must hold for \(0 \leq t \leq T\) and \(x \geq 0\).

Take the equation (73) and using equation (79) and we get
\[
\begin{align*}
e^{-rt}c(t,S(t)) = c(0,S(0)) + \int_0^t e^{-ru}c(u,(\sigma + 1)S(u - )) - c(u,S(u - ))d\tilde{M}(u)
\end{align*}
\]

(80)

In particular,
\[
\begin{align*}
e^{-rT}(S(T) - K)^+ = e^{-rT}c(T,S(T))
\end{align*}
\]

\[
\begin{align*}
= c(0,S(0)) + \int_0^T e^{-ru}c(u,(\sigma + 1)S(u - )) - c(u,S(u - ))d\tilde{M}(u)
\end{align*}
\]

(81)

Suppose we shell the call at time zero in exchange for initial capital \(X(0) = c(0,S(0))\). We want to invest in the stock and money market account so that \(X(t) = c(t,S(t))\) for all \(t\) or, equivalently,
\[
\begin{align*}
e^{-rt}X(t) = e^{-rt}c(t,S(t)) \quad \text{for all } t \in [0,T].
\end{align*}
\]

From equation (80), we see that the differential of \(e^{-rt}c(t,S(t))\) is
\[
\begin{align*}
d(e^{-rt}c(t,S(t))) &= e^{-rt} \left[ c(t,(\sigma + 1)S(t)) - c(t,S(t)) \right]d\tilde{M}(u)
\end{align*}
\]

(82)

The differential of the value \(X(t)\) of a portfolio that at each time \(t\) holds \(\Gamma(t)\) shares of stock is
\[
\begin{align*}
dX(t) &= \Gamma(t - )dS(t) + r[X(t) - \Gamma(t)S(t)]dt
\end{align*}
\]

(83)

Therefore
\[
\begin{align*}
d(e^{-rt}X(t)) &= e^{-rt} \left[ -rX(t)dt + dX(t) \right] \\
&= e^{-rt}[\Gamma(t - )dS(t) - r\Gamma(t)S(t)dt]e^{-rt}\sigma\Gamma(t - )S(t - )d\tilde{M}(t)
\end{align*}
\]

(84)

Now, we introduce in determining the value of \(\Gamma(t - )\), the position held just before any jump that may occur at time \(t\). Comparing equation (83) an (84) and we conclude that we should take
\[
\begin{align*}
\Gamma(t - ) &= \frac{c(t,(\sigma + 1)S(t - )) - c(t,S(t - ))}{\sigma S(t - )}
\end{align*}
\]

(85)

This is hedging position we should hold at all time, whether they are jumps or not. More specifically, if we define
\[ \Gamma(t) = \frac{c(t, (\sigma + 1)S(t)) - c(t, S(t))}{\sigma S(t)} \quad \text{for all } t \in [0, T] \]  

(86)

Then equation (85) will also and integration of equation (84)

\[ e^{-\tau t}X(t) = X(0) + \int_0^t e^{-\tau (u)} [c(u, (\sigma + 1)S(u)) - c(u, S(u))] d\tilde{M}(u) \]  

(87)

Comparing the equation (87) with equation (80) shows that \( X(t) = c(t, S(t)) \) for all \( t \). In particular, equation (81) shows that \( X(T) = (S(T) - K)^+ \). And the short position in the European call has been hedged.

4. Formalism of Black Scholes PDE in Jump Diffusion

The value of options is determined by supply and demand in the markets and on the other hand it is done by mathematical models. This part will explore jump diffusion models for option pricing. The Brownian motion and normal distribution have been widely used in the Black-Scholes model of option pricing to determine the return distributions of assets. Jump diffusion models always contain two parts, a jump part and a diffusion part. A common Brownian motion determines the diffusion part and a Poisson process determines the jump part.

In jump-diffusion models, a general expression for the asset price, \( S(t) \), under the physical probability measure \( P \), is given by the following stochastic differential equation

\[ \frac{dS(t)}{S(t-)} = \mu dt + \sigma dW(t) + d\left( \sum_{i=1}^{N(t)} (V_i - 1) \right) \]  

(88)

The impulse-function causes price changes in the underlying asset, and is determined by a distribution function. The jump part enables to model sudden and unexpected price jumps of the underlying asset.

Solving the SDE gives the dynamics of the asset price under the physical probability measure

\[ S(t) = S(0) e^{\left[ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right] \prod_{i=1}^{N(t)} V_i} \]  

(89)

Here \( N(t) \) is a Poisson process with rate \( \lambda \), \( W(t) \) is a standard Brownian motion and \( \mu \) is the drift rate. \( \{V_i\} \) is a sequence of independent identically distributed (i.i.d) nonnegative random variables.

In the Merton model, \( \log(V_i) = Y_i \) is the absolute asset price jump size and is normally distributed. i.e. \( \eta(x) = N(\mu, \rho) \). In the Kou model, \( \log(V_i) = Y_i \) is the absolute asset price lump size and is double exponential distributed. i.e. \( \eta(x) = \frac{p \eta_1 e^{-\eta_1 x} + q \eta_2 e^{\eta_2 x}}{1_{\{x \geq 0\}}} \).

In the Black Scholes model, the price of the underlying asset is modeled as a lognormal random variable. The stochastic differential equation (SDE) governing the dynamics of the price under the risk-neutral probability measure, is given by

\[ dS(t) = r S(t) dt + \sigma S(t) dW(t) \]  

(90)

Compared to equation (90), which is under the risk-neutral measure, \( \mu \) has taken the place of \( r \), and a Poisson process is added. The drift rate is the expected return on the stock per year. In contrast to the SDE under the risk-neutral measure, the drift component has not been adjusted for the market price of risk.

An arbitrage free option-pricing model is specified under a risk-neutral probability measure. In asset pricing, the condition of no arbitrage is equivalent to the existence of a risk-neutral measure. It arises from a key property of the Black Scholes SDE. This property is that the equation does not involve any variables that are affected by the risk preferences of investors. The SDE would not be
independent of risk preferences if it involved the expected return, $\mu$, of the stock. This is because the value of depends on risk preferences

The corresponding SDE under the risk-neutral probability measure is

$$\frac{dS}{S(t-)} = (r - \lambda k^*)dt + \sigma W(t) + d\left(\sum_{i=1}^{N^*(t)} (V_i - 1)\right)$$

(91)

Here $W^*(t)$ is a standard Brownian motion under a risk-neutral probability measure. $N^*(t)$ is a Poisson process under a risk-neutral probability measure.

Where $\lambda k^* dt$ is the expected relative price change $E\left(\frac{dS}{S(t)}\right)$ from the jump part $dN^*(t)$ in the time interval. This is the expected part of the jump. This is why the instantaneous expected return under the risk-neutral probability measure, $rdt$, is adjusted by $-\lambda k^* dt$ in the drift term of the jump-diffusion process to make the jump part an unpredictable innovation.

Solving the SDE gives the dynamics of the asset price under a risk-neutral probability measure

$$S_t = S(0)e^{\left(\left(r - \lambda k^* - \frac{\sigma^2}{2}\right) + \sigma W^*(t)\right)\prod_{i=1}^{N^*(t)} V_i^*}$$

(92)

As with any jump-diffusion model, changes in the asset’s price in the Merton model consists of a diffusion component modeled by a Brownian motion and a jump component modeled by a Poisson process. The asset price jumps are assumed to be independently and identically distributed.

The probability of a jump occurring during a time interval of length $dt$, can be expressed as

- Probability of the event does not occur in the time interval $dt = 1 - \lambda dt$
- Probability of the event occur once in the time interval $dt = \lambda dt$
- Probability of the event occur more than once in the time interval $dt = 0$

The relative price jump size, or in other words the percentage change in the asset price caused by jumps, is

$$\frac{dS_t}{S_t(0)} = \frac{y^*_i S_t - S_t}{S_t} = y^*_i - 1$$

(93)

$$\log\left(\frac{y^*_i}{1}\right) - 1 = (y^*_i - 1),$$

which is connected with equation (93)

The absolute price jump size $y^*_i$ is a nonnegative random variables drawn from a lognormal distribution, i.e. $\ln(y^*_i) = i.i.d. N(\mu, \delta^2)$. The density of the distribution is given by

$$f(y^*_i) = \frac{1}{\delta \sqrt{2\pi}} e^{-\frac{(\ln y^*_i - \mu)^2}{2\delta^2}},$$

where $\mu$ and $\delta$ are the mean and standard deviation of $y^*_i$.

This in turn implies that $E[y^*_i] = e^{\frac{\mu + \frac{1}{2}\delta^2}}$.

The relative price jump size $(y^*_i - 1)$ is log normally distributed with the mean

$$E[y^*_i - 1] = e^{\frac{\mu - \frac{1}{2}\delta^2} - 1} = k^*$$

(94)

The dynamics of the asset price, which incorporates the above properties, is given by equation (92).

There are no closed form solutions for the option price in the Merton model. However, Merton developed a solution where he specified the distribution of $Y_i$ as above, and with this, derived a solution for the price of the option. Assuming that the jumps are log-normally distributed as above, the
following expression for the price of a European call option is given in [21]. For simplicity, the superscript * is dropped.

\[ V_c(S,t) = \sum_{n=0}^{\infty} \left[ \frac{e^{-\lambda^2 \tau}(\lambda^2 \tau)^n}{n!} \right] BS_c(S,\tau=T-t,v_n^2,r_n) \]  

(95)

The \( n^{th} \) term corresponds to the scenario where \( n \) jumps occur during the life of the option. 

\[ \lambda = \lambda (1 + k) \]

\[ v_n^2 = \sigma^2 + \frac{n}{\tau} \delta^2 \]

\[ r_n = r - \lambda k + n \frac{ln(1+k)}{\tau} \]

\( \delta^2 \) is the variance of the jump diffusion and is the mean of the relative asset price jump size. 

\[ \frac{e^{-\lambda^2 \tau}(\lambda^2 \tau)^n}{n!} \] is the Poisson probability that the asset price jumps \( n \) times during the interval of length \( \tau \). Thus, the option price can be interpreted as the weighted average of the Black-Scholes price on the condition that underlying assets’ price jumps \( n \) times during the life of the option, with the weights being the probability that the assets’ price jumps \( n \) times during the life of the option.

The Kou double – exponential Jump – Diffusion Model the stock price consists of two parts. The first part is a continuous part driven by a normal geometric Brownian motion and the second part is the jump part with a logarithm of jump size, which is double exponentially distributed. The number of jumps is determined by the event times of a Poisson process.

The expression for the stock price is given by equation (88), which is under the physical probability measure.

Given that \( \log(V_i) = Y_i \) is double-exponentially distributed with the probability density function

\[ f_i(y) = p \eta_1 e^{-\eta_1 y} 1_{(y > 0)} + q \eta_2 e^{-\eta_2 y} 1_{(y < 0)} , \text{ where } \eta_1 > 1, \eta_2 > 0 \]

(96)

Where \( p, q \geq 0, p + q = 1 \) are constants and represent the physical probabilities of upwards and downward jumps. In other words,

\[ \log(V_i) = Y_i = \begin{cases} \xi^+ & \text{with probability } p \\ \xi^- & \text{with probability } q \end{cases} \]

\( \xi^+ \) and \( \xi^- \) are exponential random variables which are equal in distribution with means \( \frac{1}{\eta_1} \) and \( \frac{1}{\eta_2} \). Further, the Brownian motion and the jump process are assumed to be one-dimensional.

\[ E(Y_i) = \frac{p}{\eta_1} - \frac{q}{\eta_2} \]

\[ Var(Y_i) = pq \left( \frac{1}{\eta_1} + \frac{1}{\eta_2} \right)^2 + \left( \frac{p}{\eta_1} + \frac{q}{\eta_2} \right) \]

\[ E(V_i) = E(e^{Y_i}) = q \frac{\eta_2}{\eta_2 + 1} + p \frac{\eta_1}{\eta_1 + 1} , \text{ where } \eta_1 > 1, \eta_2 > 0 \]
The requirement $\eta > 0$ is needed to ensure that $E(V) < \infty$ and $E(S(t)) < \infty$. This essentially means that the average upward jump cannot exceed 100%, which is quite reasonable, because this is not observed in the stock market.

Conclusion
In this paper we gave a brief introduction to jump–diffusion models and review various mathematical model and use these models for option pricing. In option pricing, a jump-diffusion model is a procedure of combination model, mixing a jump process and a diffusion process. Robert C. Merton as an extension of jump models has introduced jump-diffusion models. From the above discussion, we constructed the Black - Scholes PDE with Jump-Diffusion Models. Initially, we introduced and briefly presented the stochastic models for option pricing- Jump-diffusion model. Then introduced the fundamental derivations of the Black Scholes model for the pricing of a European call option. Similarly, how we reinterpreted the Black Scholes in the formalism of jump process, in addition we considered the problem of pricing a European call when the underlying asset is jump process. Finally, we build the mathematical model for Black – Scholes PDE equation with Jump diffusion process.

References


